Chapter 2 : Reduction formulae

Dr. A. N. Bhavale

Head, Department of Mathematics,
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune-5.

12^{th} March, 2021

(I) Evaluation of $\int \frac{x^n}{\sqrt{ax^2+bx+c}}dx$ for $n \geq 0$.

Hint : Let $I_n = \int \frac{x^n}{\sqrt{ax^2+bx+c}}dx$ and note that $x^n = x \cdot x^{n-1} = \frac{(2ax+b)-b}{2a} \cdot x^{n-1}$. Therefore
$I_n = \frac{1}{2a} \int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}} \cdot x^{n-1}dx - \left(\frac{b}{2a}\right) I_{n-1}$, for $n \geq 1$.
Now using integration by parts, we get for $n \geq 2$,

$$\int x^{n-1} \cdot \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx = (2\sqrt{ax^2 + bx + c})x^{n-1}$$
$$-2a(n-1)I_n - 2b(n-1)I_{n-1} - 2c(n-1)I_{n-2},$$

by multiplication and division by $\sqrt{ax^2 + bx + c}$
Thus $I_n = \frac{\sqrt{ax^2+bx+c}}{an}x^{n-1} - \frac{b(2n-1)}{2an}I_{n-1} - \frac{c(n-1)}{an}I_{n-2}$

Remark :

Note that $I_1 = \int \frac{x}{\sqrt{ax^2+bx+c}}dx = \int \frac{(2ax+b-b)/(2a)}{\sqrt{ax^2+bx+c}}dx$
$= \frac{1}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx$
$= \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}I_0$, where $I_0 = \int \frac{1}{\sqrt{ax^2+bx+c}}dx$

Illustration 1 : Evaluate $\int \frac{x^3}{\sqrt{x^2-2x+2}}dx$.

Solution : Let $I_3 = \int \frac{x^3}{\sqrt{x^2-2x+2}}dx$.

Therefore by above reduction formula,

$I_3 = \frac{\sqrt{x^2-2x+2}}{3}x^2 - \frac{(-2)(5)}{6}I_2 - \frac{2(2)}{3}I_1$,

and $I_2 = \frac{\sqrt{x^2-2x+2}}{2}x - \frac{(-2)(3)}{4}I_1 - \frac{2(1)}{2}I_0$.

Also $I_1 = \sqrt{x^2-2x+2} + I_0$

and $I_0 = \log((x-1) + \sqrt{x^2-2x+2})$.

(II) Evaluation of $\int \frac{dx}{(x^2+a^2)^n}$ for $n \geq 1$.

Hint : Let $I_n = \int \frac{dx}{(x^2+a^2)^n}$ for $n \geq 1$.
Clearly $I_1 = \int \frac{dx}{(x^2+a^2)} = \tan^{-1}\left(\frac{x}{a}\right)$.
$\therefore$ By integration by parts, for $n \geq 2$, we get

$$I_n = \int \frac{1}{(x^2+a^2)^n} \cdot 1dx = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}$$

by adding and substracting a^2 in x^2 in the second integral.

Illustration 1 : Evaluate $\int \frac{1}{(x^2+1)^4} dx$.

Solution : Let $I_4 = \int \frac{1}{(x^2+1)^4} dx$. Here $a = 1$ and $n = 4$. Therefore by above reduction formula,

$$I_4 = \frac{x}{2(4-1)1^2(x^2+1^2)^{4-1}} + \frac{(2\times4)-3}{2(4-1)1^2} I_{4-1} = \frac{x}{6(x^2+1)^3} + \frac{5}{6} I_3.$$

Now $I_3 = \frac{x}{4(x^2+1)^2} + \frac{3}{4} I_2$ and $I_2 = \frac{x}{2(x^2+1)} + \frac{1}{2} I_1$.

(III) Evaluation of $\int (x^2 + a^2)^{n/2} dx$ for odd $n \geq 1$.

Hint : Let $I_n = \int (x^2 + a^2)^{n/2} dx$ for $n \geq 1$. Clearly $I_1 = \int \sqrt{x^2 + a^2} dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \log(x + \sqrt{x^2 + a^2})$.
$\therefore$ Using integration by parts, for $n \geq 2$, we get
$I_n = \int ((x^2 + a^2)^{n/2} \cdot 1) dx = \frac{x(x^2+a^2)^{n/2}}{n+1} + \frac{na^2}{n+1} I_{n-2}$,
by adding and substracting a^2 in x^2 in the second integral, where
$I_{n-2} = \int (x^2 + a^2)^{(n/2)-1} dx = \int (x^2 + a^2)^{((n-2)/2)} dx$.

Illustration 1 : Evaluate $\int (x^2 + 9)^{5/2} dx$.
Solution : Let $I_5 = \int (x^2 + 3^2)^{5/2} dx$. Here $a = 3$ and $n = 5$. $\therefore$ By above reduction formula,
$I_5 = \frac{x(x^2+3^2)^{5/2}}{5+1} + \frac{5(3^2)}{5+1} I_3$ and $I_3 = \frac{x(x^2+3^2)^{3/2}}{3+1} + \frac{3(3^2)}{3+1} I_1$.

(IV) Evaluation of $\int \sin^n x dx$ for $n \geq 2$.

Hint : Let $I_n = \int \sin^n x dx = \int (\sin^{n-1} x)(\sin x) dx$.
Therefore, using integration by parts and using $\cos^2 x = 1 - \sin^2 x$, we get for $n \geq 2$,
$I_n = -\frac{(\sin^{n-1} x)(\cos x)}{n} + \frac{n-1}{n} I_{n-2}$.

(V) Evaluation of $\int_0^{\pi/2} \sin^n x dx$ for $n \geq 2$.

Let $I_n = \int_0^{\pi/2} \sin^n x dx$ for $n \geq 2$.

$\therefore I_n = \left[\frac{-(\sin^{n-1} x)(\cos x)}{n} \right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} I_{n-2}$.

(VI) Evaluation of $\int_0^{\pi/2} \cos^n x dx$ for $n \geq 2$.

Let $I_n = \int_0^{\pi/2} \cos^n x dx$ for $n \geq 2$.

As $\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \cos^n(\frac{\pi}{2} - x) dx$,

$I_n = \int_0^{\pi/2} \sin^n x dx$ for $n \geq 2$.

Therefore $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$.

Note that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and $\cos(\frac{\pi}{2} - x) = \sin x$.

Case I : Suppose n is even. Then

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\left(\frac{n-7}{n-6}\right)\cdots\left(\frac{1}{2}\right) I_0,$$

where $I_0 = \frac{\pi}{2}$.

Case II : Suppose n is odd. Then

$$I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\left(\frac{n-7}{n-6}\right)\cdots\left(\frac{2}{3}\right) I_1,$$

where $I_1 = 1$.

Ex. : Evaluate $\int_0^{\pi/2} \sin^8 x dx$ and $\int_0^{\pi/2} \cos^9 x dx$.

Illustration :

Show that $\displaystyle\int_0^{\pi/6} \cos^4(3x)\sin^2(6x)dx = \frac{5\pi}{192}$

Solution : We know that $\sin 2\theta = 2\sin\theta\cos\theta$.
Now put $3x = t$. $\therefore 3dx = dt$ or $dx = \frac{dt}{3}$.
Also, when $x = 0, t = 0$ and when $x = \frac{\pi}{6}, t = \frac{\pi}{2}$.
Hence $\displaystyle\int_0^{\pi/6} \cos^4(3x)\sin^2(6x)dx = \int_0^{\pi/2} \cos^4 t(2\sin t\cos t)^2\frac{dt}{3} =$
$\displaystyle\frac{4}{3}\int_0^{\pi/2} \cos^6 t\sin^2 t dt = \frac{4}{3}\int_0^{\pi/2}(\cos^6 t - \cos^8 t)dt =$
$\displaystyle\frac{4}{3}\left(\int_0^{\pi/2}\cos^6 t dt - \int_0^{\pi/2}\cos^8 t dt\right) =$
$\displaystyle\frac{4}{3}\left(\frac{(5)(3)(1)}{(6)(4)(2)}\frac{\pi}{2} - \frac{(7)(5)(3)(1)}{(8)(6)(4)(2)}\frac{\pi}{2}\right) = \frac{5\pi}{192}$.

Thank you