Booklet I

For

Lab Course on Computational Geometry

(*Skill Enhancement Course-SEC*)

F.Y.B.Sc. (Mathematics)

BY

Dr.Mrs. Pooja M. Paratane

Department of Mathematics
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune 411005

Contents

1 Two Dimensional Transformations

Introduction

A point is the fundamental entity in geometry. Any plane figure is just a set of points of the plane satisfying the geometrical condition.Any plane figure in geometry can be represented as the set of points. For example a polygon is represented as the set of its vertices.

Definition 1.1 *A polygon is represented by a sequence of points in $XY-$ plane as $\{P_1, P_2, \cdots, P_n\}$ and is defined as the plane figure bounded by the line segments $P_1P_2,\ P_2P_3,\ P_3P_4, \cdots P_{n-1}P_n,\ P_nP_1$. The points $P_1, P_2, \cdots, P_n$ are known as vertices of polygon.*

Any point in XY-plane is specified by its co-ordinates. We represent any point by a row or column matrix.

Definition 1.2 *If $P(x, y)$ is any point in the XY-plane then a row matrix $\begin{bmatrix} x & y \end{bmatrix}_{1\times 2}$ or a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is known as position vector of the point P.*

Definition 1.3 *A point $P(x, y, z)$ in the three-dimensional space is represented by a row matrix $\begin{bmatrix} x & y & z \end{bmatrix}$ or by a column matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.*

Definition 1.4 *A polygon $P_1 \begin{bmatrix} x_1 & y_1 \end{bmatrix}, P_2 \begin{bmatrix} x_2 & y_2 \end{bmatrix}, \cdots, P_n \begin{bmatrix} x_n & y_n \end{bmatrix}$ is represented by an $n \times 2$ matrix*

$$[X] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

The matrix $[X]$ is known as the position vector of the polygon. Each row denotes the position vector of the vertex of a polygon. A geometrical transformation is always obtained by post multiplying to the position vector matrix of the given polygon.

Transformation of a point

Let P be any point in the plane with position vector $[P] = \begin{bmatrix} x & y \end{bmatrix}$ and $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a general 2×2 matrix. Consider

$$\begin{aligned} [P][T] &= \begin{bmatrix} x & y \end{bmatrix}_{1\times 2} \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \\ &= \begin{bmatrix} ax + cy & cx + dy \end{bmatrix}_{1\times 2} \\ &= \begin{bmatrix} x^* y^* \end{bmatrix} \\ &= [P^*] \end{aligned}$$

Thus the point P is transformed to point $P^* = [x^*, y^*]$ after multiplying by a matrix $[T]$ where $x^* = ax + cy$ and $y^* = bx + dy$. The matrix $[T]$ is known as the transformation matrix.

Standard Transformations: Particular cases

Sr.No.	Particular Case	Transformation Matrix $[T]$	Effect on point $[P] = [x \;\; y]$	Transformation Details
1)	$a = d = 1$, $b = c = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$[x \;\; y]$	unchanged
2)	$a \neq 0,\; d = 1$, $b = c = 0$	$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$	$[ax \;\; y]$	Scaling in x coordinate by a units
3)	$a = 1,\; d \neq 0$, $b = c = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$	$[x \;\; dy]$	Scaling in y by d units
4)	$a \neq 0,\; d \neq 0$, $b = c = 0$	$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$	$[ax \;\; dy]$	Scaling in x and y by a and d units respectively
5)	$a = d \neq 0$, $b = c = 0$	$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$	$[ax \;\; ay]$	Uniform scaling by a units
6)	$a = d = 1$, $b = 0,\; c \neq 0$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$	$[x + cy \;\; y]$	Shearing in x by c units
7)	$a = d = 10$, $b \neq 0,\; c = 0$	$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$	$[x \;\; y + bx]$	Shearing in y by b units
8)	$a = 1, d = -1, b = c = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$[x \;\; -y]$	Reflection through x-axis
9)	$a = -1, d = 1, b = c = 0$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$[-x \;\; y]$	Reflection through y-axis
10)	$a = d = 0$, $b = c = 1$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$[y \;\; x]$	Reflection through the line $y = x$
11)	$a = d = 0$, $b = c = -1$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$[-y \;\; -x]$	Reflection through the line $y = -x$

Rotation

In two dimensional plane, rotation takes place around a fixed point which is called as the center of rotation. The transformation matrix for rotation through an angle θ about origin is given as

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Conventionally the rotations are considered positive in anticlockwise sense and negative in clockwise sense.

Remark 1.1 *The determinant of the matrix of transformation for rotation* $det[T] = \cos^2\theta + \sin^2\theta = 1$.

Remark 1.2 *If we rotate the object through an angle $-\theta$ the we get back the original position of an object. The transformation matrix for rotation through angle $-\theta$ is*

$$\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Remark 1.3 *The inverse of transformation matrix* $[T]$ *is*

$$[T]^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The transpose of transformation matrix $[T]$ *is*

$$[T]' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We can observe that $[T][T]' = I_2$ which implies that $[T]^{-1} = [T]'$. Thus, the transformation matrix $[T]$ is orthogonal.
A geometrical transformation can be accomplished by post multiplying the position vector of an object by an appropriate transformation matrix.

★★★★★

Progressive Education Society's
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune - 5

Lab Course on Computational Geometry(SEC)

F.Y.B.Sc.

Practical 1

Two Dimensional Transformations

---

1. Write the transformation matrix for the following:
   (a) Scaling in x co-ordinate by 3 units.
   (b) Scaling in y co-ordinate by 2 units.
   (c) Scaling in x and y co-ordinates by 4 and -2 units respectively.
2. Write the transformation matrix for the following:
   (a) Shearing in x direction by $\frac{1}{2}$ units.
   (b) Shearing in y direction by $\frac{1}{3}$ units.
   (c) Shearing in x and y directions by $\frac{2}{3}$ and -1.5 units. respectively.
3. Write the transformation matrix for the following:
   (a) Reflection through X-axis.
   (b) Reflection through Y-axis.
   (c) Reflection through $y = x$ line.
   (d) Reflection through $y = -x$ line.
4. Write the transformation matrix for rotation through $\theta = 30°, 60°, 90°$.
5. Write down the transformation matrix $[T]$. if we want to expand the size of the cube four times the unit cube.

★★★★★

2 Transformation of Polygons

Theorem 2.1 *If the 2×2 transformation matrix transforms the points P and Q to the points P^* and Q^* respectively, then the same transformation transforms the midpoint of the line segment PQ to the midpoint of the line segment P^*Q^*.*

Theorem 2.2 *Suppose a 2×2 transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is used to transform the line segment PQ to the line segment P^*Q^*. If the slope of the line segment PQ is m, then the slope of the line segment P^*Q^* is*

$$m^* = \frac{b + dm}{a + cm}$$

Corollary 2.3 *Let PQ and RS be any two line segments. If PQ and RS are transformed to the line segments P^*Q^* and R^*S^* respectively under a general 2×2 transformation matrix, $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then PQ and RS are parallel if and only if P^*Q^* and R^*S^* are parallel.*

Theorem 2.4 *Let L be a straight line with the equation $y = mx + k$. If L is transformed to the line L^* by a 2×2 transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then equation of L^* is $y^* = m^*x^* + k^*$ where*

$$m^* = \frac{b + dm}{a + cm} \text{ and } k^* = k\left(\frac{ad - bc}{a + cm}\right).$$

★★★★★

Progressive Education Society's
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune - 5

Lab Course on Computational Geometry(SEC)

F.Y.B.Sc.

Practical 2

Transformation of Polygons

---

1. The line segment between the points $A = [2\ 3]$ and $B[-3\ 4]$ is transformed to the line segment A^*B^* using transformation matrix $[T] = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find the slope of A^*B^*.
2. If a unit square is transformed by a 2×2 transformation matrix, the resulting position vectors are $\begin{bmatrix} 0 & 0 \\ 2 & 3 \\ 8 & 4 \\ 6 & 1 \end{bmatrix}$. What was the transformation?
3. Suppose we apply shearing in y-direction by 3 units onto a line with slope 3 and y-intercept 4. Then, determine the y-intercept of the resulting line.
4. The lines $L_1 : 3x + 2y = 12$ and $L_2 : 2x - 3y + 5 = 0$ are transformed to L_1^* and L_2^* under the transformation matrix $[T] = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$. Find the point of intersection of L_1^* and L_2^*.
5. The line segment joining the points $A = [3\ 4]$ and $B = [5\ 6]$ is transformed to the line segment A^*B^* by the transformation matrix $[T] = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. Find the midpoint of A^*B^*.
6. Suppose the line $y = 2x+1$ is transformed by the 2×2 transformation matrix $[T] = \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}$. Obtain equation of the transformed line.

★★★★★

3 Combined Transformations

Many times we have to apply a sequence of the transformations. It is not possible to apply each transformation matrix separately and then get transformed object after each transformation. Instead, we can apply one combined transformation matrix on the object which is obtained by multiplying all transformation matrices applied as per their sequence.

In this process, the order in which the sequence of the transformations are applied is important. If we change the order, the result is also changed. (As the matrix multiplication is not commutative)

If a finite sequence of transformations say $[T_1], [T_2], [T_3], \cdots, [T_k]$ is to be applied on the object $[X]$, then it can be accomplished by a single transformation matrix given by

$$[T] = [T_1][T_2][T_3]\cdots[T_k]$$

$[T]$ is called the concatenated transformation matrix. Then, by applying $[T]$ on $[X]$, we can obtain $[X^*]$.

★★★★★

Progressive Education Society's
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune - 5

Lab Course on Computational Geometry(SEC)

F.Y.B.Sc.

Practical 3

Combined Transformations

---

1. Find the concatenated transformation matrix for the given sequence of transformations and apply it on the given object.

   (a) First a $82°$ rotation about origin, followed by shearing in y-direction by -1.2 units, followed by uniform scaling by factor 3.
   Apply this on the triangle with vertices $A[1\ 3]$, $B[2\ -1]$, $C[-1\ -1]$.

   (b) Scaling in both x and y co-ordinates by factors $\frac{1}{3}$ and 2 respectively, followed by reflection through the line $y = x$, followed by rotation about the origin by $-25°$.
   Apply it on the segment PQ, where $P[1.5\ -3]$, $Q[-4\ 6.5]$.

   (c) i) Shearing in x-direction by 3 units.
   ii) Scaling in y-co-ordinate by factor $\frac{3}{4}$.
   iii) Reflection through Y-axis.
   Apply it on point $P[-1\ 2]$.

   (d) i) Shearing in x and y-directions by -2.4 and 1.3 units respectively.
   ii) Rotation about origin by $45°$.
   iii) Reflection through X-axis.
   Apply it on unit square.

   (e) i) Reflection through the line $y = -x$.
   ii) Scaling in x-co-ordinates by factor $\frac{2}{3}$.
   iii) Shearing in both x and y direction by 2 and 5 units respectively.
   Apply it on segment AB where $A[1\ -2]$, $B[0\ 2]$

★★★★★

4 Solid Body Transformations

Area of Transformed Figure:

Any plane figure can be approximated by adjoining unit squares and hence it is a transformed figure by adjoining parallelogram. Hence, the same relationship holds between the areas of any plane figure and its transformed figure.

If A_i denotes the area of the initial figure and A_t denotes area of transformed figure, then

$$A_t = A_i \det[T]$$

Solid Body Transformation:

The transformation $[T]$ is said to br solid body transformation if it preserves magnitude and the angle between two lines.

The transformation matrix $[T]$ is solid body transformation matrix if $[T]$ is an orthogonal matrix determinant 1.

$$\text{i.e } [T][T]^t = I_2$$
$$\text{i.e } [T]^{-1} = [T]^t \text{ and } \det[T] = 1.$$

★★★★★

Progressive Education Society's
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune - 5

Lab Course on Computational Geometry(SEC)

F.Y.B.Sc.

Practical 4

Solid Body Transformations

---

1. If we apply transformation matrix $[T] = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ on a square, then we get a parallelogram of area 64 cm^2. Find the length of each side of the original square.
2. If we apply shearing in x and y directions by -2 and 2 units respectively onto the rectangle of length 20 cm, then it results into the parallelogram of area 1500 cm^2. Find the breadth of the rectangle.
3. Determine whether the transformation matrix $[T] = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$ is solid body transformation.
4. A rectangle of length 7 units and breadth 5 units is transformed under the transformation matrix $[T] = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$. Find the area of the transformed figure.
5. If the circle of circumference 14π units is uniformly scaled by 3 units. What is the area of the transformed circle.
6. Determine if the transformation $[T] = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{-3}{5} \end{bmatrix}$ is solid body transformation.
7. If a 2×2 transformation matrix $[T] = \begin{bmatrix} 2 & -3 \\ 2 & -1 \end{bmatrix}$ is applied to a circle of radius 3 cm, then find the area of the resulting figure.
8. if a square with sides 2 cm is reflected through y-axis, then what is the area of transformed figure?
9. Determine if the transformation of reflection through y-axis is a solid body transformation. justify your answer.
10. Determine if rotation about origin through 68° is a solid body transformation. Justify your answer.

★★★★★

5 Homogeneous Co-ordinates and Translation

The transformation of scaling, reflection, rotation, shearing are accomplished by a 2×2 transformation matrix.

The origin is invariant under any 2×2 transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The object cannot be shifted from one place to other place due to this.

It can be overcomed by limitation of a 2×2 transformation matrix.

For this, we need to introduce homogeneous co-ordinates.

Homogeneous Co-ordinates:

Let P be any point in the XY-plane with co-ordinates (x, y). The homogeneous co-ordinates of the point P are given as (x', y', h), where $x = \dfrac{x'}{h}$, $y = \dfrac{y'}{h}$ and 'h' is any real number.

The homogeneous co-ordinates of point P is represented by 1×3 matrix $\begin{bmatrix} x' & y' & h \end{bmatrix}$. This matrix is known as homogeneous co-ordinate position vector of P.

for example, Let $P = (-2, 5)$ Homogeneous co-ordinate position vector of P for different values of h are

$$P\begin{bmatrix} -2 & 5 \end{bmatrix} \xrightarrow[homo.]{h=1} \begin{bmatrix} -2 & 5 & 1 \end{bmatrix} \qquad h = 1$$

$$P\begin{bmatrix} -2 & 5 \end{bmatrix} \xrightarrow{h=2} \begin{bmatrix} -4 & 10 & 2 \end{bmatrix} \qquad h = 2$$

$$P\begin{bmatrix} -2 & 5 \end{bmatrix} \xrightarrow{h=\frac{1}{2}} \begin{bmatrix} -1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \qquad h = \tfrac{1}{2}$$

Any point in XY-plane is not represented in a unique way by homogenoues co-ordinate position vector.

Note that the set of homogeneous co-ordinates are $\begin{bmatrix} x & y & 1 \end{bmatrix}$ (here $h = 1$). This gives the physical position vector (x, y)

The transformation matrix to transform a point with homogeneous co-ordinates is given as

$$[T] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The homogeneous co-ordinate position vector P is $[P] = \begin{bmatrix} x & y & 1 \end{bmatrix}$

Transforming point $[P]$, we get

$$\begin{aligned} [P][T] &= \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + cy & bx + dy & 1 \end{bmatrix} \\ &= \begin{bmatrix} x^* & y^* & 1 \end{bmatrix} \\ &= [P^*] \end{aligned}$$

It follows that the entries a, b, c and d of the upper left 2×2 submatrix have the same effects as 2×2 transformation matrix

i.e they produce effects scaling, shearing, reflection, rotation.

Translation:

Consider the transformation matrix $[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}$

Applying $[T]$ on a point $P = \begin{bmatrix} x & y & 1 \end{bmatrix}$

$$\begin{aligned} [P][T] &= \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} \\ &= \begin{bmatrix} x+m & y+n & 1 \end{bmatrix} \\ &= \begin{bmatrix} x^* & y^* & 1 \end{bmatrix} \end{aligned}$$

$$x^* = x + m,\ y^* = y + n$$

This effect is known as translation by m and n units in X and Y directions respectively.

★★★★★

Progressive Education Society's
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune - 5

Lab Course on Computational Geometry(SEC)

F.Y.B.Sc.

Practical 5

Homogeneous Co-ordinates and Translation

---

1. Translate the point $P\begin{bmatrix}2 & -6\end{bmatrix}$ in Y-direction by 4 units.
2. Translate the point $A\begin{bmatrix}-1 & 3\end{bmatrix}$ in X-direction by -3 units.
3. Translate the point $Q\begin{bmatrix}3 & -2\end{bmatrix}$ so that it coincide origin.
4. Shift the origin to the point $P\begin{bmatrix}4 & -6\end{bmatrix}$.
5. Find the concatenated transformation matrix for translation in X-direction by 4 units, followed by rotation about the origin through an angle $-60°$, followed by reflection through Y-axis. Apply it on the line segment between the point $A\begin{bmatrix}2 & -1\end{bmatrix}$ and $B\begin{bmatrix}1 & -3\end{bmatrix}$.
6. Find the concatenated transformation matrix for the following sequence of transformations:
   (a) Reflection through the line $y = x$.
   (b) Shearing in x and y directions by -3 and 2 units.
   (c) Translation in x and y directions by 1 and -2 units respectively.
   Apply it to triangle $\triangle ABC$, where $A\begin{bmatrix}2 & 4\end{bmatrix}$, $B\begin{bmatrix}4 & 6\end{bmatrix}$, $C\begin{bmatrix}2 & 6\end{bmatrix}$.
7. Obtain the concatenated transformation matrix that first reflects about the line $x = 0$, second translate by -1 in both x and y directions, third rotates about the origin by $180°$.
   Apply it on the point $P\begin{bmatrix}5 & 2\end{bmatrix}$.
8. Reduce the size by 4, then apply rotation about origin through $32°$ then translate in x and y directions by -3 and 3 units respectively.
   Apply the concatenated transformation matrix of above sequence on a rectangle given by $O\begin{bmatrix}0 & 0\end{bmatrix}$, $A\begin{bmatrix}2 & 0\end{bmatrix}$, $B\begin{bmatrix}2 & 3\end{bmatrix}$, $D\begin{bmatrix}0 & 3\end{bmatrix}$.

★★★★★