Lab Manual

Subject: Lab Course on Analytical Geometry

For

Under Graduate Students

BY

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Preface

This laboratory manual on Analytical Geometry is designed to complement the theoretical understanding of the subject with hands-on computational and graphical activities. It is intended for undergraduate students pursuing mathematics, engineering, or related disciplines, and aims to strengthen their geometric intuition, analytical skills, and problem-solving techniques through practical exercises.

Analytical Geometry serves as a foundational pillar in mathematics, linking algebra and geometry through the use of coordinate systems and equations. It plays a vital role in various applications across science, engineering, computer graphics, architecture, and physics. This manual covers core topics such as lines, planes, conic sections, and three-dimensional geometry, with a balanced blend of conceptual explanation and practical experimentation.

Each experiment in the manual includes a clear objective, theoretical background, step-by-step procedure, and a set of observation and conclusion sections. Where applicable, software tools like GeoGebra, MATLAB, Python, or other graphical utilities may be incorporated to enhance the learning experience and promote visual understanding. Students are encouraged to use this as a tool for both guided instruction and self-directed learning.

I hope this manual serves as a valuable resource for both students and instructors and contribute meaningfully to the teaching and learning of Analytical Geometry.Taking these sample questions instructors can design the different examples. Constructive feedback and suggestions for improvement are always welcome.

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Contents

1 Analytical Geometry of Two Dimensions

A] Equations of standard conics:

1. **Circle:-**

   (a) **Standard circle:-** $x^2 + y^2 + r^2$
   centre = (0,0), radius = r

   (b) **General circle:-** $(x - h)^2 + (y - k)^2 = r^2$
   centre = (h, k), radius = r

   (c) It can be generalized as,
   $x^2 + y^2 + 2gx + 2fy + c = 0$
   centre = $(-g, -f)$, radius = r = $\sqrt{g^2 + b^2 - c}$

2. **Ellipse :-**
   Equation: $\dfrac{x^2}{d^2} + \dfrac{y^2}{b^2} = 1$
   centre = (0,0),
   semimajor axis = a;   semiminor axis = b
   focii: $(ae, 0), (-ae, 0)$
   eccentricity = e ,
   Remember relation: $b^2 = a^2(1 - e^2)$

3. **Hyperbola:-**
   Equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
   centre = (0,0)
   conjugate axis = 2a
   transverse axis = 2b
   Remember relation : $b^2 = a^2(e^2 - 1)$

4. **Parabola:-**
   Equation: $y^2 = 4ax$
   symmetry = about x-axis
   focus = $(a, 0)$
   vertex = (0,0)
   length of latus rectum = 4a
   Equation of directrix : x + a = 0

Note:-

1. These curves are section of cone by plane. Hence we call it as conics.

2. The pair of straight lines is the conic section The intersection of cone with the plane passing through the vertex of cone consist of two straight line(called generator of cone)

3. A vertex of cone is a circle of radius. O is a point. Thus a point is also a conic section.

4. All conic sections are governed by a general equation of second degree in x and y
   viz. $ax^2 + 2hxy + by^2 + 2qx + 2fy + c = 0$ (1)

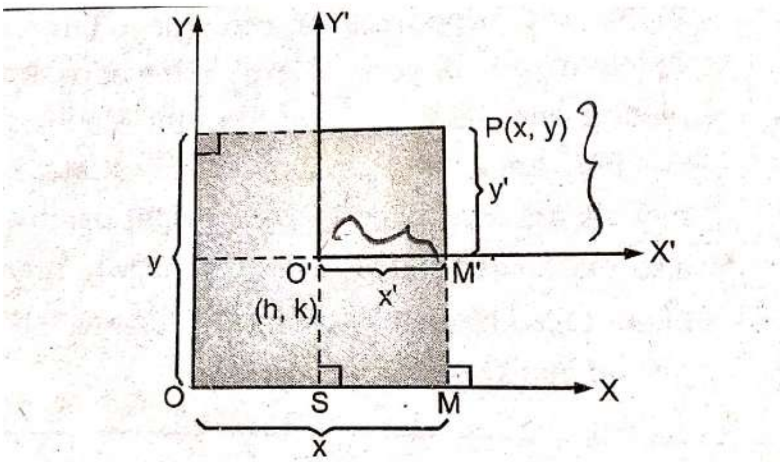
5. The nature and the position of conic depends upon the values of the coefficients appearing in above equation (1).

6. To reduce equation (1), we have to transform the original coordinate system to new coordinate system. This can be done by two types of transformation of original coordinate system.
   (a) Translation of Axes (b) Rotation of Axes.

B] Transformation of axes

I)Translation of axes:-

In Translation of axes , the origin is shifted to new point in plane. The direction of new axes remains unchanged. The new coordinate axes are parallel to original coordinate axes.



original centre = O = (0,0)
new centre = $O' = (h, k)$
Let $P(x, y)$ be any point in original coordinate system. Let $P(x', y')$ be new coordinates of the point in new coordinate system, when origin is shifted to new origin $O'(h, k)$.
Then , the relation between original and new coordinates of point P is

$$x = x' + h$$
$$y = y' + k$$

II) Rotation of Axes:-

In rotation of an axes , the origin remains unchanged , but both axes are rotated through an angle θ.

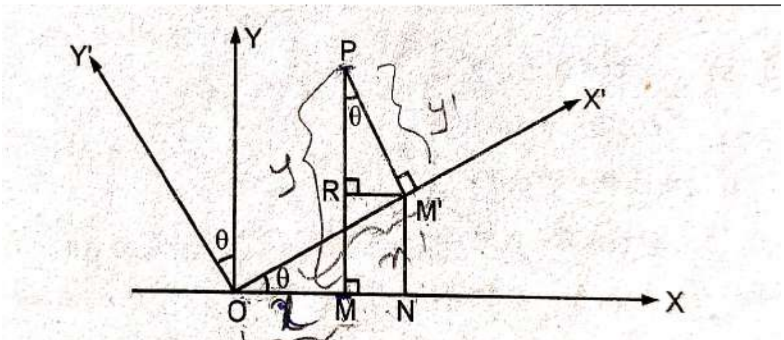


Fig. 5.2

Let the new coordinates of point $P(x, y)$ be $P(x', y')$. Suppose the coordinate axes are rotated through an angle θ, then the relation between original coordinates and new coordinates of point P is given as ,

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

This transformation can be expressed in matrix form as

$$[x, y] = [x', y'] \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

C] Central Conics

Center of conic: A point in the plane of conic which bisect every chord of a conic passing through it , is called the centre of the conic.

Central conic : A conic having a centre is called central conic.
1) Circle , Hyperbola , Ellipse are central conics
2) Parabola is not a central conics. Parabola has a vertex.
3) The product term xy is absent always in equation of central conics.

Discrimination of conics and Reduction Procedure:

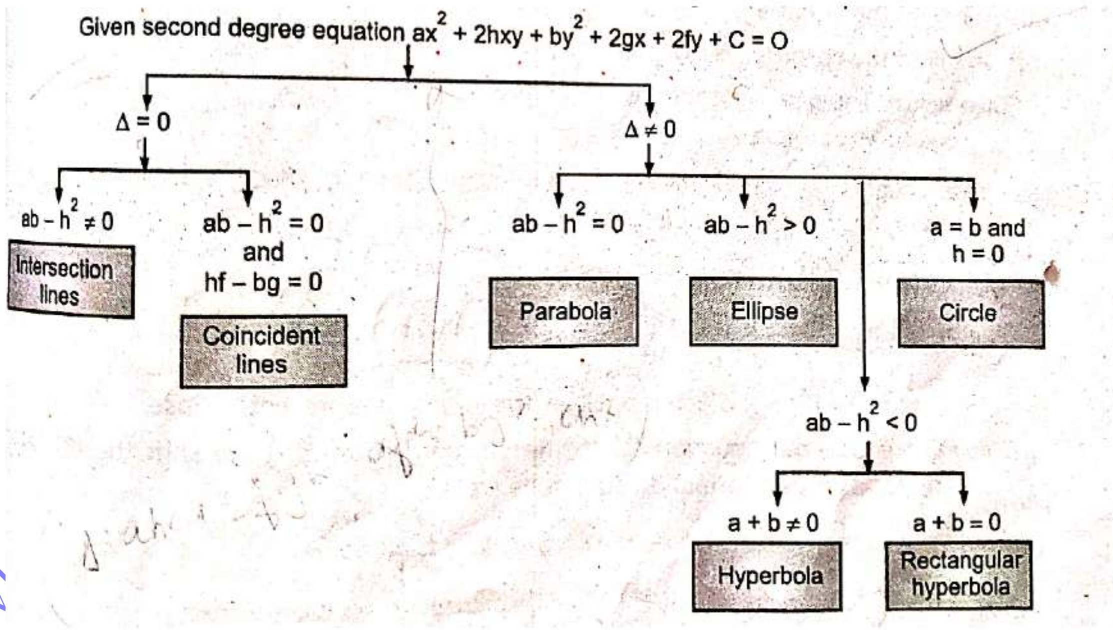
The general second degree equation represent the conics :-
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)

Step 1: Given second degree equation is given by equation (1).
Find value of constants a, h, b, g, f, and c fron equation.

Step 2: Equation (1) represent central conics if ,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

Step 3: Nature of conic:
Based on value of $ab - h^2$, we can discriminate nature of a conics
It can be shown diagrammatically below



Step 4: Let the centre of conic be (α, β)

Then the coordinate of centre of conic can be obtained by solving equation

$$a\alpha + h\beta + g = 0$$
$$h\alpha + b\beta + f = 0$$

solving these equation we get,

$$(\alpha, \beta) = \left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$$

Step 5: Reduce the equation by shifting origin to new centre (α,β)

The equation of transformation are :

$$x = x' + \alpha$$
$$y = y' + \beta$$

The equation of conics at new centre $(\alpha, \beta)$is given as ,

$$ax'^2 + 2hx'y' + by'^2 + \left(\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}\right) = 0$$

$$\Rightarrow ax'^2 + 2hx'y' + by'^2 + \frac{\Delta}{ab - h^2} = 0 \qquad (2)$$

We can use equation (2) directly as equation of conic obtained by shifting origin to new centre (α, β)

Step 6: To remove product term from equation (2) , we have to rotate axes through an angle θ, where

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a - b}\right)$$

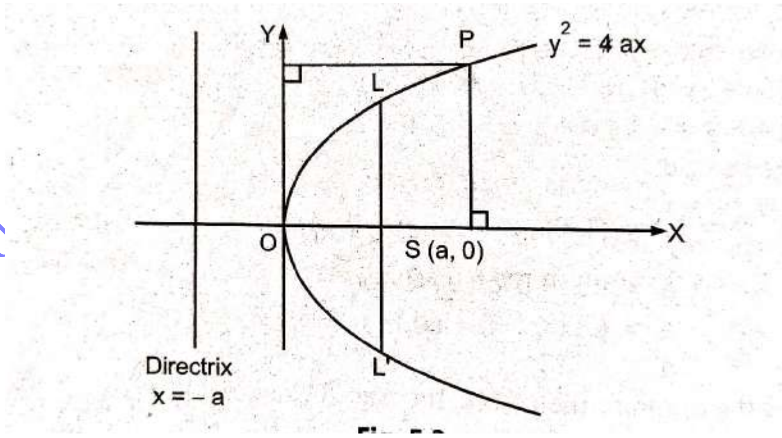
The equations of transformation are :

$$x' = x" \cos\theta - y" \sin\theta$$
$$y' = x" \sin\theta - y" \cos\theta$$

Step 7: Substitute value of x' and y' in equation (2) and after simplifying we can obtain the standard equation of conic.

D] General Equation representing parabola:

General equation of parabola: $y^2 = 4ax$



Procedure:

Let P(x, y) be any point on parabola.
Axis of parabola is x-axis (y = 0) and tangent to the parabola is y-axis.
Then, we have
x = perpendicular distance of P from tangent to parabola
y = perpendicular distance of P from axis of parabola
Consider the general equation of second degree: $ax^2+2hxy+by^2+2gx+2fy+c=0$
for which $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, but $ab - h^2 = 0$ so that the equation represents the parabola.
Since $ab - h^2 = 0$, then the second degree terms in (1) forms a perfect square. i.e

$$ax^2 + 2hxy + by^2 = (lx + my)^2$$
$$(1) \quad \Rightarrow (lx + my)^2 + 2gx + 2fy + c = 0$$
$$\Rightarrow (lx + my)^2 = -[2gx + 2fy + c] \qquad (3)$$

If the lines $lx+my = 0$ and $2gx+2fy+c = 0$ are perpendicular to each other, then the line $lx+my = 0$ is considered as the **axis of parabola** and $2gx+2fy+c = 0$ is taken as **the tangent to the parabola at the vertex.**

If the lines $lx + my = 0$ and $2gx + 2fy + c = 0$ are not perpendicular to each other, first we have to make them perpendicular by adding λ to the equation of lines $lx + my + \lambda = 0$ and $2gx + 2fy + c = 0$.

Adding λ in equation (3),

$$(lx + my + \lambda)^2 - 2lx\lambda - 2my\lambda - \lambda^2 = -[2gx + 2fy + c] \qquad (4)$$
$$\Rightarrow (lx + my + \lambda)^2 = 2lx\lambda + 2my\lambda + \lambda^2 - 2gx - 2fy - c$$

$\Rightarrow (lx + my + \lambda)^2 = (2l\lambda - 2g)x + (2m\lambda - 2f)y + (\lambda^2 - c)$

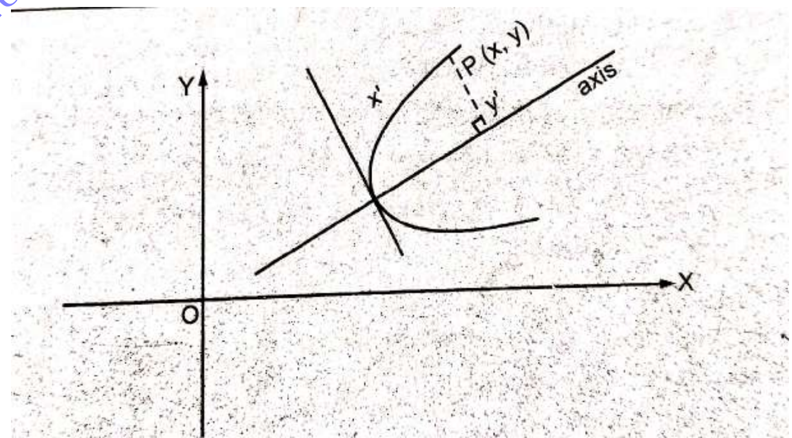
We have to find λ s.t the lines

$$lx + my + \lambda = 0$$
$$\text{and} \quad (2l\lambda - 2g)x + (2m\lambda - 2f)y + (\lambda^2 - c) = 0$$

will be perpendicular to each other.
i.e s.t

$$\left(-\frac{l}{m}\right)\left(-\frac{2l\lambda - 2g}{2m\lambda - 2f}\right) = -1$$

Substituting value of λ so determined, we get the equation of the axis of parabola i.e $y' = 0$ as the lines $lx + my + \lambda = 0$ and the tangent to the parabola i.e $x' = 0$ as $(2l\lambda - 2g)x + (2m\lambda - 2f)y + (\lambda^2 - c) = 0$



Let P(x, y) be coordinate of point P in original coordinate system
Then ,
x = ⊥ distance of P from tangent to parabola
y = ⊥ distance of P from axis to parabola

$$x' = \frac{(2l\lambda - 2g)x + (2m\lambda - 2f)y + (\lambda^2 - c)}{\sqrt{(2l\lambda - 2g)^2 + (2m\lambda - 2f)^2}}$$
$\Rightarrow (2l\lambda - 2g)x + (2m\lambda - 2f)y + (\lambda^2 - c) = (\sqrt{(2l\lambda - 2g)^2 + (2m\lambda - 2f)^2})x'$

$$y' = \frac{lx + my + \lambda}{\sqrt{(l)^2 + (m)^2}}$$
$\Rightarrow lx + my + \lambda = (\sqrt{(l)^2 + (m)^2})y'$
Substituting in (3) we get standard equation of parabola.

E] Summary:

- Let $P(x, y)$ be any point in the XY plane with origin $O(0, 0)$. If origin is shifted to $O'(h, k)$ then the co-ordinates (x', y') with respect to origin O' are given by , $x' = x - h,\ y' = y - k$
- Rotation of axes: Let $P(x, y)$ be any point in the XY- plane. If the axes are rotated through an angle θ, then the coordinate (x', y') of P with respect to rotated axes are given by $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$.

- If the axes are rotated through an angle θ so that the transformed form of the expression $ax^2 + 2hxy + by^2$ is free from the product term xy, then $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$.

- If under the rotation of axes, without shifting the origin ,the expression $ax^2 + 2hxy + by^2$ is transformed to $a'x'^2 + 2hx'y' + by'^2$ then $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$

- The general second degree equation of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents conic and $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$ is the centre of conic .

- In the general equation of conic

  1. If $ab - h^2$ =0 then it represents parabola.
  2. If $ab - h^2$ >0 then it represents ellipse.
  3. If $ab - h^2$ <0 then it represents hyperbola.
  4. If $a = b$ and $h = 0$ then it represents circle.
  5. If $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ then it represents pair of straight lines .

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1. The origin is changed to the point $(-2, k)$. Determine the value of k so that the new equation of locus given by $2y^2 + 3x + 4y = 0$ will not contain first degree term in y.

2. Under the translation of axes, the expression $2x^2 - 3y^2 + 4y + 5$ is transformed into $2x'^2 - 3y'^2 + 4x' - 8y' + 3$. Find the co-ordinates of new origin with respect to old origin.

3. Find the form of the equation $2x^2 + 3xy - 4y^2 + x + 3 = 0$ when origin is shifted to the point $(-2, 1)$.

4. Transform the equation $3x^2 - 2xy + 3y^2 + 8x + 3y + 4 = 0$ by rotating the axes through an angle θ where $\theta = \sin^{-1}\left(\frac{3}{5}\right)$, $0 < \theta < \frac{\pi}{2}$.

5. Find the transformed form of equation $x^2 + 4xy + y^2 = 0$ when the axes are rotated through an angle $\theta = \tan^{-1}(3)$ wihout changing the origin.

6. Find the angle θ through which the axes should be rotated to remove the xy term in the folowing equations:

   (a) $x^2 - 4xy + 4y^2 - 2y + 2 = 0$.

   (b) $7x^2 + 12xy - 5y^2 + 4x + 3y - 5 = 0$.

   (c) $4x^2 + 12xy + 9y^2 + 2x + 2y + 7 = 0$.

   (d) $8x^2 - 12xy + 7y^2 + 4x + 6y - 2 = 0$.

   (e) $3x^2 - 5xy + 3y^2 - 5 = 0$.

   (f) $4x^2 + 2\sqrt{3}xy + 2y^2 - 7 = 0$.

   (g) $5x^2 + 3xy + y^2 + x - y - 2 = 0$.

   (h) $4x^2 + 6xy + 4y^2 - 2x + 2y + 3 = 0$.

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Practical 2: Equations of Conics

Ques: Reduce the following equations to its standard form. Also find the center/vertex of conic.

1. $5x^2 + 6xy + 5y^2 - 10x - 6y - 3 = 0.$
2. $7x^2 - 6xy + 7y^2 + 30x + 10y + 35 = 0.$
3. $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0.$
4. $3x^2 - 8xy - 3y^2 - 10x - 4y + 2 = 0.$
5. $x^2 - 2xy + y^2 - 6x - 2y + 4 = 0.$
6. $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0.$
7. $x^2 - 4xy - 2y^2 + 10x + 4y = 0.$
8. $x^2 + 12xy - 4y^2 - 6x + 4y + 9 = 0.$
9. $9x^2 - 6xy + y^2 - 14x - 2y + 12 = 0.$
10. $x^2 - xy + 2y^2 - 2x - 6y + 7 = 0.$

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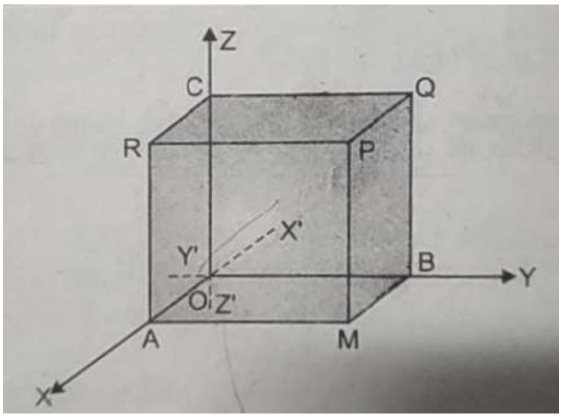
2 Three Dimensional System

A] Introduction of Three dimensional space

Consider the following figure of three dimensional system.



1. Three coordinate axes X, Y and Z-axes.
2. Three coordinate axes divide the whole space into eight parts called as octant.
3. Three main coordinate planes XOY, YOZ and XOZ-planes
4. Signs of coordinates of point are governed by the octant in which that point lies as given in the following table.

Octant	x	y	z
OXYZ	+	+	+
OXYZ'	+	+	−
OXY'Z	+	−	+
OXY'Z'	+	−	−
OX'YZ	−	+	+
OX'YZ'	−	+	−
OX'Y'Z	−	−	+
OX'Y'Z'	−	−	−

B] Distance Formula

If $P(x_1, y_1, z_1) and Q(x_2, y_2, z_2)$ are two points in space then the distance between then is

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

C] Section Formula

Internal Division:

Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ and $R(x, y, z)$ be a point on PQ which divides the segment PQ internally in the ratio $m : n$. Then the co-ordinates of point

R is given as

$$x = \frac{mx_2 + nx_1}{m+n}$$
$$y = \frac{my_2 + ny_1}{m+n}$$
$$z = \frac{mz_2 + nz_1}{m+n}$$

External Division:

Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ and $R(x, y, z)$ be a point on PQ which divides the segment PQ externally in the ratio $m : n$. Then the co-ordinates of R is given as

$$x = \frac{mx_2 - nx_1}{m-n}$$
$$y = \frac{my_2 - ny_1}{m-n}$$
$$z = \frac{mz_2 - nz_1}{m-n}$$

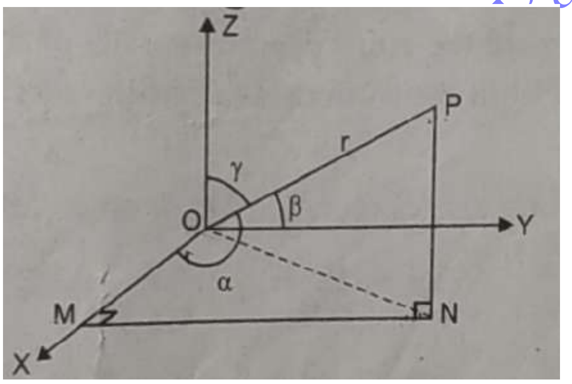
D] Directions Cosines

Direction angles :- Let AB be a directed line in the space. If line AB makes angles α, β and γ with the positive directions of X, Y and Z axes respectively, then the angles α, β and γ are called direction angles.

Direction cosines :- If α, β and γ are the direction angles made by the line AB with the positive directions of X, Y and Z axes then cosine ratios of direction angles α, β, γ viz. $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines (d.cs) of the line AB. These are denoted by $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$.

Axis	Direction angles	D.Cs
X-axis	0°, 90°, 90°	1, 0, 0
Y-axis	90°, 0°, 90°	0, 1, 0
Z-axis	90°, 90°, 0°	0, 0, 1

Theorem 1 :- Let L be a line in a space with direction cosines l, m, n, then $l^2 + m^2 + n^2 = 1$



Consider the line L in the space with direction cosines l, m, n. Let $P(x, y, z)$ be any point on the line L.
Let the line OP makes angles α, β, γ with positive directions of $X, Y, Z-$ axes respectively. Then

$$l = \cos\alpha,\ \ m = \cos\beta,\ \ n = \cos\gamma$$
$$\text{Let } OP = r = \sqrt{x^2 + y^2 + z^2}$$

Draw a perpendicular from P on XY plane. Let the foot of perpendicular be the point N. Also, draw a perpendicular from point N on $X-$axis such that $NM \perp X-$axis
Then $OM = x$, $\ MN = y$ and $PN = z$
Join ON.
Now, $PN \perp ON$
$\therefore \triangle ONP$ is right angled triangle at N
Now $\angle POZ = \gamma$; $\angle PON = 90^0 - \gamma$

In $\triangle ONP$,

$$\sin\angle PON = \frac{PN}{OP}$$
$$\sin(90^0 - \gamma) = \frac{z}{r}$$
$$\implies \cos\gamma = \frac{z}{r}$$

Similarly,we can get In $\triangle ONP$,

$$\cos\alpha = \frac{x}{r}$$
$$\cos\beta = \frac{y}{r}$$

Consider ,

$$\begin{aligned} l^2 + m^2 + n^2 &= \cos^2\alpha + \cos^2\beta + \cos^2\gamma \\ &= \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \\ &= \frac{x^2 + y^2 + z^2}{r^2} \\ &= \frac{r^2}{r^2} = 1 \\ &= l^2 + m^2 + n^2 \\ &= 1 \end{aligned}$$

E] Direction Ratios

If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are actual direction cosine of a line then any three number which are proportional to them are called direction ratios (d.rs) of a line.
Suppose the numbers a, b, c are d.rs of line having d.cs $\cos\alpha$, $\cos\beta$, $\cos\gamma$ i.e

$l, m, n.$
Then we have

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c} \qquad \Rightarrow \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

$$\text{Each ratio} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \pm\frac{1}{\sqrt{a^2+b^2+c^2}} \qquad (\text{since} \quad l^2+m^2+n^2 = 1)$$

$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$$

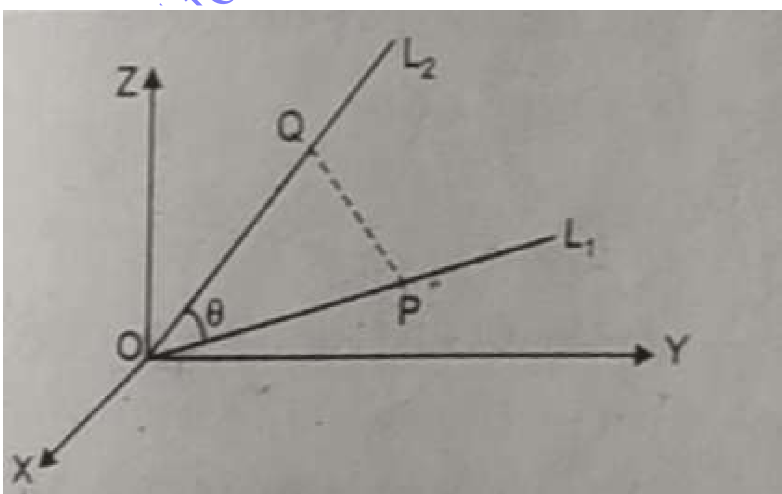
$$m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

This gives the relation between d.rs and d.cs of a line.

F] Angle between two lines

Theorem 2 :- The cosine of angle between lines L_1 and L_2 having d.cs l_1, m_1, n_1 and l_2, m_2, n_2 respectively is given by, $\cos\theta = l_1l_2 + mm_2 + n_1n_2$
Proof :- Consider the lines L_1 and L_2 in the space
Given d.cs of L_1 aa l_1, m_1, n_1 and d.cs of L_2 are l_2, m_2, n_2
Choose point on $P(x_1, y_1, z_1)$ on line L_1 such that $l(OP) = 1$ and $Q(x_2, y_2, z_2)$ on line L_2 such that $l(OQ) = 1$



Let θ be angle between the lines L_1 and L_2.
Now let OP make direction angles $\alpha_1, \beta_1, \gamma_1$ and OQ make angles $\alpha_2, \beta_2, \gamma_2$ with positive direction of $X, Y, Z-$ axis respectively.
Thus we have ,
$\cos\alpha_1 = \frac{x}{OP} \implies \cos\alpha_1 = x_1$ (Refer proof of Theorem 1 to understand

this. Here $r = 1$)

$\cos\beta_1 = \dfrac{y}{OP} \implies \cos\beta_1 = y_1$

$\cos\gamma_1 = \dfrac{z}{OP} \implies \cos\gamma_1 = z_1$

Thus, coordinates of point P are given by

$(x_1, y_1, z_1) = (\cos\alpha_1, \cos\beta_1, \cos\gamma_1) = (l_1, m_1, n_1)$

Similarly, coordinates of point Q are given by

$(x_2, y_2, z_2) = (\cos\alpha_2, \cos\beta_2, \cos\gamma_2) = (l_2, m_2, n_2)$

In $\triangle OPQ$, by cosine Rule , we have

$$\begin{aligned}\cos\theta &= \frac{(OP)^2 + (OQ)^2 + (PQ)^2}{2(OP)(OQ)} \\ &= \frac{1 + 1 - (PQ)^2}{2(1)(1)} \\ &= \frac{2 - (PQ)^2}{2} \qquad (1)\end{aligned}$$

Now , by distance formula,

$$\begin{aligned}(PQ)^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ &= [(\cos\alpha_1 - \cos\alpha_2)^2 + (\cos\beta_1 - \cos\beta_2)^2 + (\cos\gamma_1 - \cos\gamma_2)^2] \\ (PQ)^2 &= (\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1) + (\cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2) \\ &\quad - 2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2) \\ (PQ)^2 &= 1 + 1 - 2(l_1l_2 + m_1m_2 + n_1n_2) \\ &= 2 - 2(l_1l_2 + m_1m_2 + n_1n_2)\end{aligned}$$

Substituting value of $(PQ)^2$ in (1) we get,

$$\begin{aligned}\cos\theta &= \frac{2 - [2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2} \\ &= \frac{2 - 2 + 2(l_1l_2 + m_1m_2 + n_1n_2)}{2} \\ \cos\theta &= l_1l_2 + m_1m_2 + n_1n_2\end{aligned}$$

Result 1 :-Sine of angle between two lines

We have,

$$\begin{aligned}\sin^2\theta &= 1 - \cos^2\theta \\ &= 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2 \\ &= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2 \\ &\text{since} \quad (l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_2^2 + m_2^2 + n_2^2 = 1) \\ \Rightarrow \sin^2\theta &= l_1^2l_2^2 + l_1^2m_2^2 + l_1^2n_2^2 + m_1^2l_2^2 + m_1^2m_2^2 + m_1^2n_2^2 + n_1^2l_2^2 + n_1^2m_2^2 + n_1^2n_2^2 \\ &\quad - [l_1^2l_2^2 + m_1^2m_2^2 + n_1^2n_2^2 + 2l_1l_2m_1m_2 + 2m_1m_2n_1n_2 + 2l_1l_2n_1n_2] \\ &= (l_1^2m_2^2 + m_1^2l_2^2 - 2m_1m_2l_1l_2) + (l_1^2n_2^2 + n_1^2l_2^2 - 2l_1l_2n_1n_2) \\ &\quad + (m_1^2n_2^2 + n_1^2m_2^2 - 2m_1m_2n_1n_2) \\ &= (l_1m_2 - m_1l_2)^2 + (l_1n_2 - n_1l_2)^2 + (m_1n_2 - n_1m_2)^2 \\ \therefore \sin\theta &= \pm\sqrt{(l_1m_2 - m_1l_2)^2 + (l_1n_2 - n_1l_2)^2 + (m_1n_2 - n_1m_2)^2}\end{aligned}$$

Result 2:- Condition of perpendicularity
Two lines are perpendicular to each other if and only if $\theta = 90^o$
i.e if $\cos\theta = \cos 90^o = 0$
i.e if $l_1l_2 + m_1m_2 + n_1n_2 = 0$
This is the condition of perpendicular lines

Result 3 :- Condition for two perpendicular lines in the form of direction ratios
If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively
We know,

$$\begin{aligned}\cos\theta &= l_1l_2 + m_1m_2 + n_1n_2\\ &= \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\\ &+ \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\\ &+ \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\\ &= \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\end{aligned}$$

Two lines are perpendicular if $\theta = 90^o$
i.e $\cos\theta = \cos 90^o = 0$
i.e if $\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = 0$
i.e if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
This is the condition of perpendicular lines in the form of d.rs of lines

Result 4 :- Condition for parallel lines
Two lines are parallel to each other, then we consider $\theta = 0^o$
i.e if $\sin\theta = \sin 0^o = 0$
i.e if $\sqrt{(l_1m_2 - m_1l_2)^2 + (l_1n_2 - n_1l_2)^2 + (m_1n_2 - n_1m_2)^2} = 0$
$\Rightarrow (l_1m_2 - m_1l_2)^2 + (l_1n_2 - n_1l_2)^2 + (m_1n_2 - n_1m_2)^2 = 0$
$\Rightarrow l_1m_2 - m_1l_2 = 0$
$\Rightarrow l_1n_2 - n_1l_2 = 0$
$\Rightarrow m_1n_2 - n_1m_2 = 0$
$\Rightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2}; \quad \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}; \quad \dfrac{l_1}{l_2} = \dfrac{n_1}{n_2}$
$\Rightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$
These is the condition of parallel lines

Result 5 :-
Two lines having d.rs a_1, b_1, c_1 and a_2, b_2, c_2 are parallel to each other if ,
$\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

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Practical 3: Direction Ratios ans Direction Cosines

1. A line makes angles $45^o, 60^o$ with X, Y-axes. Find the angle made by the line with Z-axis.
2. Find the direction cosines of a line whose direction ratios are $6, -2, 3$.
3. Find direction cosines of a line which are equally inclined with the co-ordinate axes.
4. Find direction cosines of a line passing through the points $(2, 3, -1)$ and $(0, -1, 2)$.
5. Find the angle between two lines whose direction cosines are connected by relations $2l + 2m - n = 0$ and $lm + mn + ln = 0$.
6. Can the numbers $\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ be direction cosines of a line? Justify your answer.
7. If $\alpha,\ \beta,\ \gamma$ are direction angles of a line, then find value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

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3 The Plane

General equation of First Degree

A surface is called a plane, if a line joining any two points on the surface completely lies on the surface. i.e. if A and B are any points on the surface and P is any point on the line AB, then P also lies on the surface.

Theorem 3.1 *Every equation of first degree in x, y, z represents a plane.*

1. General equation of the plane is $ax + by + cz + d = 0$. The coefficients a, b, c represents direction ratios of normal to the plane.

2. **Point-Normal Form:** The equation of the plane passing through the point (x_1, y_1, z_1) and having d.rs a, b, c is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

3. **Normal form:** The normal form of equation of plane is $lx + my + nz = p$ where l, m, n are the direction cosines of normal to the plane and p is the prependicular distance of the plane from the origin or length of normal from the origin. The distnce p is given as

$$p = \left| \frac{d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

4. **Plane Passing Through Three Points:**
The equation of plane passing through the three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

5. **Intercept Form:** If the plane intersect the coordinate axes at the points $(a, 0, 0), (0, b, 0), (0, 0, c)$ then it's equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

6. **Perpendicular distance** of a point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by,

$$\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

7. **Angle between two Planes:** Angle between two planes is equal to angle between their normals from any point.
Let

$$a_1x + b_1y + c_1z + d_1 = 0 \quad (1)$$
$$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0 \quad (2)$$

be equations of two planes which intersects each other. The numbers a_1, b_1, c_1 and a_2, b_2, c_2 are direction ratios of normals to planes (1) and (2) respectively.

Let θ be the angle between the planes and hence between their normals. Then, we have

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

8. **Bisecting Planes:** Let

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ \text{and} \quad a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}$$

be equations of two planes which intersects each other.
Let $P(x, y, z)$ be the point on the plane bisecting angle between given planes. Then the perpendicular distances from P to the two given planes should be equal. Therefore

$$\left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Thus, equations of the bisecting planes are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Of these two bisecting planes, one bisects the acute angle and other obtuse angle between the two given planes.

9. **System of Plane:** Consider equations of two planes

$$\begin{aligned} u \equiv a_1x + b_1y + c_1z + d_1 &= 0 \\ \text{and} \quad v \equiv a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}$$

Then, equation of plane passing through the line of intersection of two planes $u \equiv 0$ and $v \equiv 0$ is given by

$$u + \lambda v = 0 \text{ , where } \lambda \text{ is real parameter.}$$

10. **Joint Equation of Two Planes:** Consider equations of two planes

$$\begin{aligned} u \equiv a_1x + b_1y + c_1z + d_1 &= 0 \\ \text{and} \quad v \equiv a_2x + b_2y + c_2z + d_2 &= 0. \end{aligned}$$

Then the joint equation of these two planes is given by

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0$$

Theorem 3.2 *The necessary and sufficient condition that the homogeneous second degree equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ respresents two planes is*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Remark 3.3 *If θ is angle between the planes reoresented by the equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, then*

$$\tan\theta = \frac{2\sqrt{(f^2 + g^2 + h^2 - ab - bc - ca)}}{a + b + c}$$

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Practical 4: Equation of Plane

Ques. Find the equation of plane

1. making intercepts $2,\ -3,\ 4$ on the coordinate axes.
2. passing through the point $(1,-2,3)$ and parallel to the plane $2x+y-z=5$.
3. passing through the points $(2,0,-1),\ (1,2,1)$ and $(-3,1,-2)$.
4. points $(1,1,0)$, $(-1,3,4)$ and is perpendicular to the plane $x+y-2z+3=0$.
5. passing through $(2,0,3)$ and makes intercepts on the axes which are in the ratio $3:\ 1:\ 2$.
6. passing through the points $A(1,1,1)$,$(1,-1,1)$ and $(-7,-3,-5)$.

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Practical 5: Bisecting Planes

1. Find angle between the planes $2x - 2y + 4z = 1$ and $x + 2y + z = 3$.
2. Find the distance of a point $(1, -1, 1)$ from the plane $x + 2y - 2z + 3 = 0$.
3. Find equation of the plane bisecting the angle between planes $x+y-z+3 = 0$ and $4x-3y+12z+13 = 0$. State which plane bisects acute angle and obtuse angle.
4. Find equation of the plane bisects the acute angle between planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z - 3 = 0$.
5. Show that the plane $14x - 8y + 13 = 0$ bisects the obtuse angle between the planes $3x + 4y - 5z + 1 = 0$ and $5x + 12y - 13z = 0$.

Practical 6: System of Planes

1. Find the equation of the plane which passes through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and is parallel to the line whose direction ratios are $2, 4, 5$.

2. Find the equation of plane containing the line of intersection of the planes $4x+3y+2z+2 = 0$ and $3x+2y+2z+1 = 0$ whose distance from the origin is $\frac{1}{\sqrt{2}}$.

3. Find the equation of plane containing the line of intersection of the planes $x - y + 2z + 1 = 0$ and $2x + y - z - 5 = 0$ and passing through the point $(1, -2, 3)$.

4. Determine which of the following equations represents pairs of planes, if so, find angle between each pair.

   (a) $12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$

   (b) $2x^2 - 2y^2 + 4z^2 + 6xz + 2yz + 3xy = 0$

   (c) $3x^2 - 6xy - 10yz + 5zx + x - 2y = 0$

   (d) $x^2 + y^2 + z^2 + 2xy + 4yz + 2zx + x - 2y + 1 = 0$

   (e) $2x^2 - 2y^2 - 3z^2 + 3xy + 5yz - 5zx + 4x - 7y + 9z - 6 = 0$

5. Find the joint equation of the planes $2x + 3y - z = 0$ and $x - y + 5z = 0$.

6. Find the equation of plane containing the line of intersection of the planes $x+y-z+1 = 0$ and $2x-y+2z+3 = 0$ at a distance unity from the origin.

7. Find the equation of plane containing the line of intersection of the planes $x - 3y + 4z = 5$ and $2x + y + z = 1$ and which is perpendicular to the plane $x + y - z = 2$.

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4 Line

1. **Equations of a Line:**

   The intersection of two planes is a line. Thus, the eqautions of two planes simultaneously represents the line. Consider two intersecting planes

   $$a_1x + b_1y + c_1z + d_1 = 0 \quad (3)$$
   $$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0 \quad (4)$$

   The line L is represented by the equations of planes given by (3) and (4)

2. **Symmetrical form of equations of a Line :**

   A] If l, m, n are direction ratios of a line and (x_1, y_1, z_1) is any point on the line then symmetric equations of line are

   $$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

   B] If $a, b, c$are direction ratios of a line and (x_1, y_1, z_1) is any point on the line then the symmetric equation of line is

   $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

3. The equations of a line given by

   $$a_1x + b_1y + c_1z + d_1 = 0$$
   $$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0$$

   is called unsymmetrical form of equation of line.

4. **Reduction of unsymmetrical form to symmetrical form:**
   Consider the line given by:

   $$a_1x + b_1y + c_1z + d_1 = 0$$
   $$\text{and} \quad a_2x + b_2y + c_2z + d_2 = 0$$

   (a) **For the point on the line:**

   A point on the line can be chosen conveniently, where the line cuts the XOY plane. i.e. $z = 0$ plane. Substitute $z = 0$ in both equations and solve that simultaneous equations to get the point on a line.

   (b) **For the direction ratios of the line:**

   The line of intersection is common to both planes. Therefore, line is perpendicular to the normals of planes. Using the condition of perpendicular lines, we get direction ratios of the line of intersection.

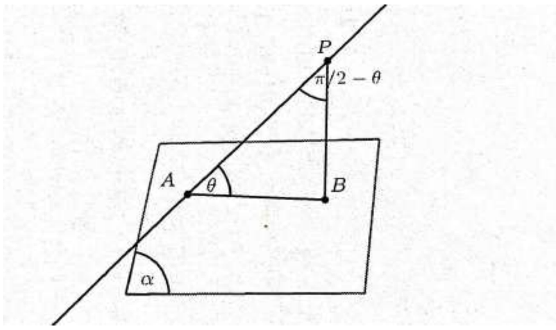
5. **Line passing through two points:** The equation of line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

6. The equation of line involves four arbitrary constants.

7. **Angle between Line and Plane:**

Let θ be angle between a line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and a plane $ax + by + cz + d = 0$.



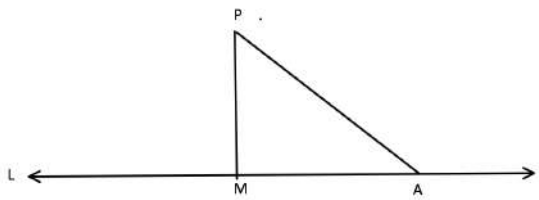
$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}}$$
$$\sin\theta = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}}$$

If a', b', c' are direction ratios (d.r.s) of the line, angle between the line and plane is given as:

$$\sin\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

8. **Length of the perpendicular from a point to a line:**

Point: $P(\alpha, \beta, \gamma)$ and the line: $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$



Step 1:- Let $PM \perp Line$, M is foot of perpendicular from P on the line.

Step 2:- The general point on the line is $(x_1 + ar, y_1 + br, z_1 + cr) = M$ (say)

Step 3:- Direction ratios of the line PM are $(x_1+ar-\alpha, y_1+br-\beta, z_1+cr-\gamma)$

Step 4:- By condition of perpendicular lines, we have

$$a(x_1+ar-\alpha)+b(y_1+br-\beta)+c(z_1+cr-\gamma)=0$$

Step 5:- Solving equation in *step 4*, we will get value of r as

$$r=\frac{a(x_1-\alpha)+b(y_1-\beta)+c(z_1-\gamma)}{a^2+b^2+c^2}$$

.

Step 6:- Substituting value of r in M, we get coordinates of point M and $d(PM)$

$$d(PM)=\sqrt{(x_1+ar-\alpha)^2+(y_1+br-\beta)^2+(z_1+cr-\gamma)^2}$$

.

9. **Condition for two lines to be coplanar:**

Consider two lines

$$L_1: \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$
$$\text{and } \; L_2: \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

are coplanar if

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of lines.

If direction ratios of lines are given as a_1, b_1, c_1 and a_2, b_2, c_2, then the condition of coplanarity is

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

The **equation of plane containing** coplanar lines is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

10. **Skew Lines:**

Two lines in the space are said to be skew line if that lines are neither intersecting nor parallel.

11. **Shortest distance between the skew lines:**

Consider two skew lines as

$$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$
$$\text{and} \quad L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

The shortest distance between lines L_1 and L_2 is given by

$$S.D = \frac{\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(a_1c_2-a_2c_1)^2+(b_1a_2-b_2a_1)^2}}$$

The **equation of a line of the shortest distance** between the skew lines is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ b_1c_2-b_2c_1 & a_1c_2-a_2c_1 & b_1a_2-b_2a_1 \end{vmatrix} = 0$$

OR

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_2 & b_2 & c_2 \\ b_1c_2-b_2c_1 & a_1c_2-a_2c_1 & b_1a_2-b_2a_1 \end{vmatrix} = 0$$

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Practical 7: Lines

1. Find the equations of a line joining the points $(-2, 1, 3)$ and $(3, 1, -2)$.

2. Find the equation of a line passing through $(3, 1, 2)$ and prependicular to the plane $2x - 2y + z + 3 = 0$. Also, find the coordinates of the foot of perpendicular.

3. Find co-ordinates of the point on the line corresponding to parameter t.
   Given Line: $\dfrac{x+1}{1} = \dfrac{y-2}{-1} = \dfrac{z+4}{1}$ at $t = -1, 2$.

4. Find the distance of point $(-2, 1, 5)$ from the line which passes through the point $(2, 3, 5)$ and has direction ratio $(2, -3, 6)$.

5. Find the distance of point $(6, 6, -1)$ from the line $\dfrac{x-2}{1} = \dfrac{y-1}{2} = \dfrac{z+3}{-1}$.

6. Find the distance of the point $(1, -1, 2)$ from the line $\dfrac{x}{2} = \dfrac{y-1}{-2} = \dfrac{z+2}{-2}$.
   Also find the coordinates of foot of perpendicular.

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Practical 8: Asymmetric Equation of line

1. Find the symmetric form of the equation of a line $x + y + z + 1 = 0$ and $4x + y - 2z + 2 = 0$.
2. Find equation of the line of intersection of two planes $x + 4y - 5z = 12$ and $8x + 12y - 13z = 32$.
3. Find the point of intersection of a line $x + y + 4z + 3 = 0 = x + 2y - z - 7$ with $YOZ-$ plane.
4. Find the direction cosines of a line given by $x + 2y - 3z = 4$ and $2x + y + z = 2$.
5. Find a symmetric equations of line given by $3x + 2y + z - 5 = 0$ and $x + y + z + 5 = 0$.
6. Find angle between lines $x + y + 2z - 33 = 0 = 2x + y + z + 1$ and $\dfrac{x-1}{2} = \dfrac{y}{1} = \dfrac{z-2}{-1}$.

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1. Find the distance of point $P(1, -2, \frac{1}{2})$ from the point of intersection of the plane $x + y + 2z = 4$ with the line $x = y - 1 = 2z + 3$.

2. Find the angle between a line $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and the normal to a plane $2x - y - z + 3 = 0$. Find also the point of intersection of line and plane.

3. Find the distance of a point $(-7, -4, 3)$ on the line $\frac{x-4}{11} = \frac{y+3}{1} = \frac{z-1}{-2}$ measured in the direction of a line from a plane $x + 3y - z = 5$.

4. Find the angle between a plane $x + 2y + z - 3 = 0$ and a line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$.

5. Find the point of intersection of the line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-3}{-2}$ with plane $3x + 4y + 5z = 5$.

6. Show that a plane $2x - y + 3z = 6$ contains the line $\frac{x-4}{3} = \frac{y+7}{-6} = \frac{z+3}{-4}$.

7. Find the equation of a plane containing the given line $x - 4 = 2y + 1 = 3z - 2$, and point $(1, -1, 3)$.

8. Find the equation of a plane containing the given line $\frac{x+2}{2} = \frac{y+3}{3} = \frac{z-4}{-2}$ and point $(0, 6, 0)$.

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Practical 10: Coplanar and Skew Lines

1. Show that the lines $\frac{x-8}{1}=\frac{y-5}{3}=\frac{z-6}{2}$ and $\frac{x-4}{11}=\frac{y+3}{1}=\frac{z-1}{-2}$ are coplanar. Find the equation of the plane containing them. Also find the point of intersection.

2. Show that the lines $\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}$ and $\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$ are coplanar. Hence find the equation of the plane passing through these lines.

3. Find the shortesrt distance between the lines

   (a) $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}, \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

   (b) $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}, \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$

   (c) $\frac{x-2}{5}=\frac{y+1}{-2}=\frac{z-2}{-3}, \frac{x-1}{-3}=\frac{y-1}{2}=\frac{z-1}{1}$

   (d) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{3}, \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$

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5 Sphere

Definition 5.1 *A sphere is a locus of a point which remains at a constant distance from a fixed point. The constant distance is called the radius and the fixed point is called the centre of a sphere.*

1. **Equation of a Sphere:**

   (a) **Equation of standard Sphere:** Equation of standard sphere with centre at origin $(0,0,0)$ and radius r is given as

   $$x^2+y^2+z^2=r^2 \tag{5}$$

   (b) **Centre Radius Form of Sphere:** Equation of sphere with centre at $C(a,b,c)$ and radius r is

   $$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \tag{6}$$

   Remark 5.1 *(a) The equation (6) can be written in the form*

   $$\begin{aligned} &x^2-2ax+a^2+y^2-2by+b^2+z^2-2cz+c^2=r^2\\ &x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (7)\\ \textit{where}\ &u=-a,\ v=-b,\ w=-c\ ,d=a^2+b^2+c^2-r^2. \end{aligned}$$

   Equation (7) is called as general equation of sphere.

   (b) The following are the characteristics of the general equation (7)

   i. It is the second degree equation in $x,\ y,\ z$;

   ii. The coefficient of $x^2,\ y^2,\ z^2$ are all equal;

   iii. The product terms $xy,\ yz,\ zx$ are absent.

   (c) The equation of the sphere contains four independent constants.

   (d) Four non-coplanar points determine the unique sphere.

2. The centre and radius of sphere represented by genral equation (7) is given as

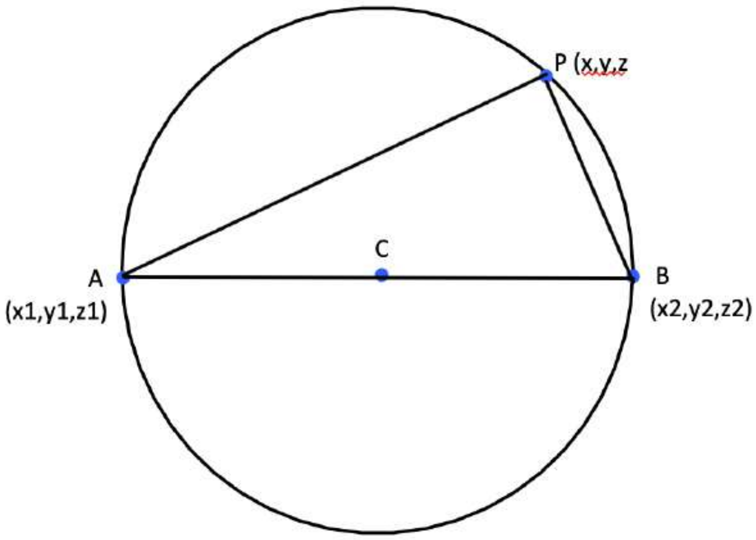
   $$\begin{aligned}\text{Centre} &= C=(-u,-v,-w)\\ \text{Radius} &= r=\sqrt{u^2+v^2+w^2-d}\end{aligned}$$

3. **Equation of the sphere passing through the four points** $A(x_1,y_1,z_1), B(x_2,y_2,z_2), C(x_3,$ and $D(x_4,y_4,z_4)$ is given by

   $$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1\\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1\\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1\\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1\\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$$

4. **Sphere with given diameter:**

Consider the sphere with diameter AB where $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. Let $P(x, y, z)$ be any point on the sphere.



Consider the sphere with diameter AB where $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. Let $P(x, y, z)$ be any point on the sphere.
$\therefore PA \perp PB$
Direction ratios of PA are $x - x_1, y - y_1, z - z_1$ and that of PB are $x - x_2, y - y_2, z - z_2$. Since PA is perpendicular to PB ,

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

This is diameter form of equation of sphere.

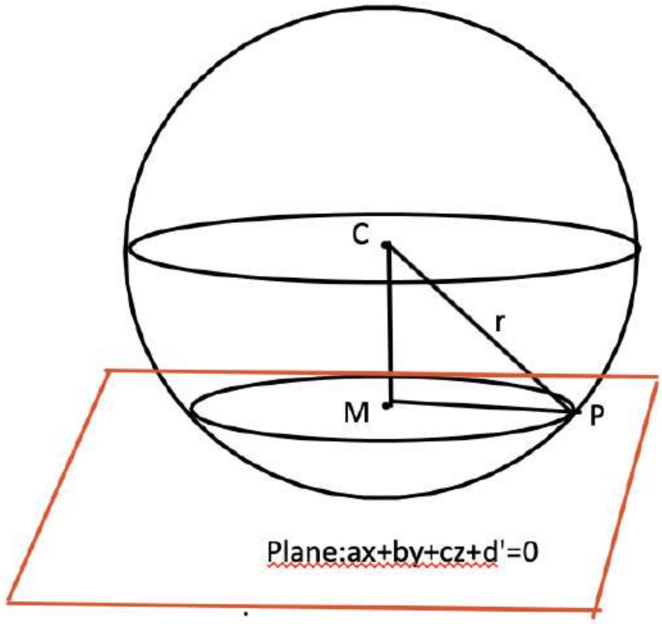
5. **Plane Section of a Sphere:**

The section of a sphere by plane is a circle.
The circle is given by

$$S : x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (8)$$
$$\text{and } U : ax + by + cz + d' = 0 \quad (9)$$

Steps to obtain the centre and radius of the circle of intersection of sphere given by equations (8) and (9).

(a) Consider the circle given by equations (8) and (9).
Centre of sphere S is $C = (-u, -v, -w)$ and radius $= r = \sqrt{u^2 + v^2 + w^2 - d}$
Direction ratios of normal to plane U are a, b, c.



(b) **To find radius of circle:**

Let M = centre of circle of intersection

MP = Radius of circle of intersection

CM = the perpendicular distance of centre C from the plane.

$$\therefore \quad CM = \left| \frac{-au - bv - cw - d'}{\sqrt{a^2 + b^2 + c^2}} \right|.$$

$$MP^2 = CP^2 - CM^2 = r^2 - \left| \frac{-au - bv - cw - d'}{\sqrt{a^2 + b^2 + c^2}} \right|^2$$

(c) **To find co-ordinates of centre of circle**

i. CM is normal to the plane. $\therefore$ d.r.s of CM are a, b, c

ii. Thus, equation of line CM passing through point $C = (-u, -v, -w)$ are given as

$$\frac{x+u}{a} = \frac{y+v}{b} = \frac{z+w}{c} = t \text{ (say)}$$

iii. M is point on the line CM.
$\therefore$ Co-ordinates of point $M = (at - u, bt - v, ct - w)$.

iv. Point M lies on the plane. Hence coordinates of point M satisfies equation of the plane.

v. Thus, $a(at-u)+b(bt-v)+c(ct-w)+d' = 0$. Solving this equation, we get value of t. Substituting it in $M = (at - u, bt - v, ct - w)$, we get centre of circle of intersection.

6. **Intersection of two Spheres:**

Consider the spheres

$$S_1 : x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \tag{10}$$
$$S_2 : x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \tag{11}$$

The curves of intersection of two spheres given by (10) and (11) is a circle.

7. **Sphere through given circle:**

   Equation of the sphere through the circle of intersection of the sphere $S : x^2+y^2+z^2+2ux+2vy+2wz+d = 0$ and the plane $U : ax+by+cz+d' = 0$ is given by

   $$S + \lambda U = 0 \text{ where } \lambda \text{ is the real parameter.}$$

8. **Intersection of Sphere and Line:**

   A line intersects the sphere in two points.

   Points of the intersection of the sphere:
   Consider the sphere $S : x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$
   and the line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = r$ (say)
   The coordinates of general point P on the line are $P(\alpha + lr, \beta + mr, \gamma + nr)$
   If the point P lies on sphere then we have

   $$\begin{aligned} &(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + lr) \\ &+ 2v(\beta + mr) + 2w(\gamma + nr) + d = 0 \\ \Longrightarrow\ &r^2(l^2 + m^2 + n^2) + 2r(\alpha l + \beta m + \gamma n + nl + vm + wn) \\ &+ (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \end{aligned}$$

   This is quadratic equation in r and hence it gives two values of r say r_1 and r_2. Therefore, there are two points as $A(\alpha + lr_1, \beta + mr_1, \gamma + nr_1)$ and $B(\alpha + lr_2, \beta + mr_2, \gamma + nr_2)$

9. **Equation of the tangent plane to standard sphere** $S : x^2+y^2+z^2 = a^2$ at point $P(x_1, y_1, z_1)$ is

   $$xx_1 + yy_1 + zz_1 = a^2.$$

10. **Equation of the tangent plane to general sphere** $S : x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at point $P(x_1, y_1, z_1)$ is

    $$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

11. **Condition of Tangency for standard sphere:**

    The condition that the plane $lx + my + nz = p$ is tangent to the sphere $x^2 + y^2 + z^2 = a^2$ is

    $$p = \pm a\sqrt{l^2 + m^2 + n^2}$$

12. **Condition of Tangency for general sphere:**

The condition that the plane $lx + my + nz = p$ is tangent to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is

$$(lu + mv + nw + p)^2 = (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2)$$

Remark 5.2 *Let C be centre and r be radius of the sphere and P be any point in the space. If*

(a) $CP < r$, then the point P lies inside the sphere.

(b) $CP = r$, then the point P lies on the sphere.

(c) $CP > r$, then the point P lies outside the sphere.

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Practical 11: Equation of Sphere

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1. Find the centre and radius of the sphere $x^2 + y^2 + z^2 + 6x - 4y + 2z + 5 = 0$.
2. Find the equation of the sphere :
   (a) Passing through $(1, 1, 2)$ and $(0, -2, 1)$ and the center lies on the line $x - 1 = 2 - y = z + 1$.
   (b) Passing through the origin and making equal positive intercept of 2 units of the axes.
   (c) Which circumscribe the tetrahedron $(0, 0, 0), (0, 3, 0), (1, 0, 0), (0, 0, 2)$.
3. Find the equation of sphere whose diameter has end points $(-2, 3, 5)$ and $(1, -2, 3)$.
4. Find the equation of smallest sphere passing through $(2, 3, -5)$ and $(1, 2, 2)$.
5. Find the equation of the sphere passing through points $(0, 0, 0)$,$(1, 0, 0)$,$(1, 1, 0)$ and $(1, 1, 1)$.
6. Find the equation of the sphere passing through points $(1, 1, 1)$,$(-1, 0, 2)$,$(0, 3, -1)$ and $(2, 1, -3)$.

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Practical 12: Plane Section of Sphere

1. Find the centre and the radius of circle $x^2 + y^2 + z^2 - 4x + 6z - 3 = 0, x + 2y - 2z = 17$.
2. Find the area and circumference of the circle represented by $x^2 + y^2 + z^2 + 2x + 3y - 6 = 0; x - 2y + 4z - 9 = 0$.
3. Find the point where the line $x + y - 3 = 0 = 2x + z + 4$ cuts the sphere $x^2 + y^2 + z^2 - 4z - 9 = 0$.
4. Find coordinates of the points where the line $\dfrac{x+3}{4} = \dfrac{y+4}{3} = \dfrac{z-8}{-5}$ intersects the sphere $x^2 + y^2 + z^2 + 2x - 10y = 23$.
5. Find the length of intercept cutoff by the line $x + 2 = y + 3 = z + 5$ on the sphere $x^2 + y^2 + z^2 - 6x + 5z + 11 = 0$.
6. Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 + 2x + 3y - 6 = 0; x - 2y + 4z - 9 = 0$ and through the centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.
7. Find the equation of sphere for which the circle $x^2 + y^2 + z^2 - 4x - 6y - 12 = 0, x + 2y - 2z + 1 = 0$ is a great circle.
8. Obtain equation of the circle lying on the sphere $x^2+y^2+z^2-2x+2y-4z+3 = 0$ and having centre $(2, 2, -3)$.
9. Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point $(1, 2, 3)$.
10. Find the equation of the sphere passing through the circle of intersection of $x^2 + y^2 + z^2 + 6x - 4y - 6z - 14 = 0$, $x + y - z = 0$ and the point $(1, 1, -1)$.

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Practical 13: Tangent Plane

1. Show that the line $\dfrac{x+7}{4} = \dfrac{y-6}{-5} = \dfrac{z+4}{5}$ is a tangent line to the sphere $x^2 + y^2 + z^2 - 4x - 6y + 2z - 19 = 0$. Also , find the point of contact.

2. Find the equation of a tangent plane to the sphere $x^2 + y^2 + z^2 + 4x - 2y + 2z - 12 = 0$ at $(2, 2, 0)$.

3. Find the value of λ if the plane $x + y + z = \lambda$ touches the sphere $x^2 + y^2 + z^2 + 2x + 2y - 2z - 9 = 0$.

4. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 4x - 2y + 2z - 12 = 0$ which are parallel to the plane $4x + y + z = 5$. Find their point of contact.

5. Show that the plane $U : 2x - 2y + z + 16 = 0$ is tangent plane to the sphere $S : x^2 + y^2 + z^2 + 2x - 4y + 2z - 3 = 0$ and also find the point of contact.

6. Determine whether the plane $2y - 3z + 18 = 0$ is tangent to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 2z - 7 = 0$.

7. Find the equation of the sphere which pass through the circle $x^2+y^2+z^2 = 5$, $x + 2y + 3z = 3$ and touches the plane $3x - 4z - 34 = 0$.

8. Show that the spheres $x^2 + y^2 + z^2 - 4x - 2y - 4z + 5 = 0$ and $x^2 + y^2 + z^2 - 6x - 6y + 17 = 0$ touch each other and find their point of contact.

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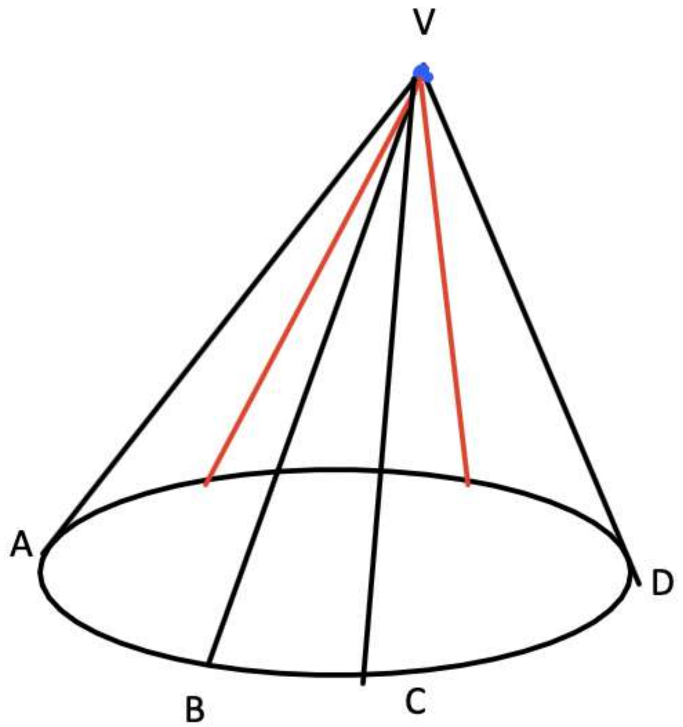
6 Cones and Cylinders

6.1 Cone

Definition 6.1 *A surface generated by a stright line passing through a fixed point and intersecting a given curve is called as a cone.*
The fixed point is called the vertex of the ***cone*** *and the given curve is called the* ***guiding curve****. A line which generates the cone is called a* ***generator****.*



Cone with guiding curve

Remark 6.1 *If the guiding curve is a plane curve of degree n, then the equation of the cone is also of degree n and we call it a cone of order n.*

Equation of Cone

To find the equation of cone with vertex at the point (α, β, γ) and having guiding curve a conic $f(x, y) = 0, z = 0$ in the $XY-$ plane.
Let $V(\alpha, \beta, \gamma)$ be the vertex of cone. The equations of a generating line passing through V are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{12}$$

At a point of this line in the $XY-$plane , we have $z = 0$.

$\therefore$, we get

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{\gamma}{n}.$$
$$x = \alpha - \left(\frac{l}{n}\right)\gamma$$
$$y = \beta - \left(\frac{m}{n}\right)\gamma \quad \text{at } P$$

Now, the point P with these coordinates lies on the conic $f(x,y) = 0$

$$\therefore \quad f\left(\alpha - \left(\frac{l}{n}\right)\gamma, \beta - \left(\frac{m}{n}\right)\gamma\right) = 0 \tag{13}$$

We eliminate l, m, n from equations (12) and (13)

$$\therefore \quad f\left(\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma, \beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right) = 0$$
$$f\left(\frac{\alpha z - \gamma x}{z-\gamma}, \frac{\beta z - \gamma y}{z-\gamma}\right) = 0 \tag{14}$$

Equation (14) is required equation of the cone.

Condition that represents a cone

Consider the general equation of second degree

$$ax^2 + cy^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2ux + 2vy + 2wz + d = 0 \tag{15}$$

Equation (15) represents the cone if

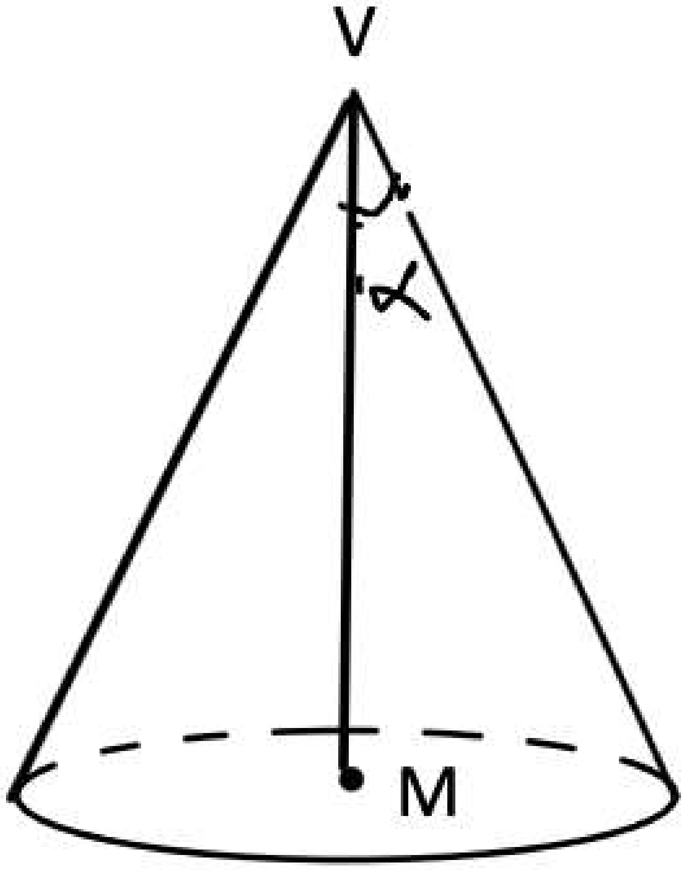
$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

If this condition is satisfied then the coordinates of the vertex $V(x_1, y_1, z_1)$ are obtained by solving the following system of simulteneous equations

$$ax_1 + hy_1 + gz_1 + u = 0$$
$$hx_1 + by_1 + fz_1 + v = 0$$
$$gx_1 + fy_1 + cz_1 + w = 0$$

Right Circular Cone:

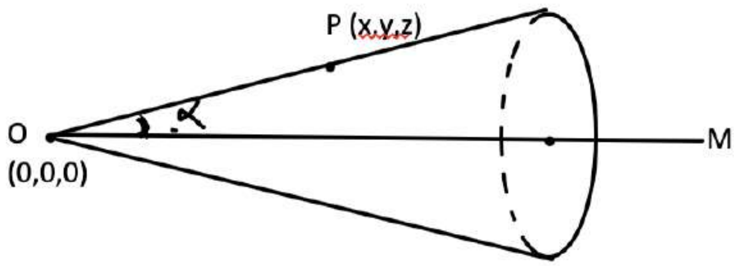
A surface generated by a line, passing through a fixed point, and making constant angle with a fixed line through that fixed point is called a right circular cone. The fixed point is vertex of the cone. The fixed line is called axis and the constant angle is called semivertical angle of the cone.
Consider the following figure of right circular cone with vetex V, axis VM and semivertical angle α.



Note: The section of a right circular cone by a plane perpendicular to the axis of the cone is a circle

Equation of Right Circular Cone:

Consider the right circular cone having vertex at origin $V(0,0,0)$, the axis $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, (where l, m, n are direction cosines of axis) and semiverticle angle α.



OM is axis of cone and α is a semivericle angle.
The equations of axis OM are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Let P be any point on the cone having coordinates (x, y, z) on the cone. Then, the direction ratios of OP are $x - 0, y - 0, z - 0$ i.e x, y, z.
The direction cosines of OP are then

$$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \cos\alpha = (l)\frac{x}{\sqrt{x^2+y^2+z^2}} + (m)\frac{y}{\sqrt{x^2+y^2+z^2}}, +(n)\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \cos\alpha = \frac{lx+my+nz}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow (lx+my+nz)^2 = (x^2+y^2+z^2)\cos^2\alpha \tag{16}$$

This is the required equation of a right circular cone.

Particular Case:

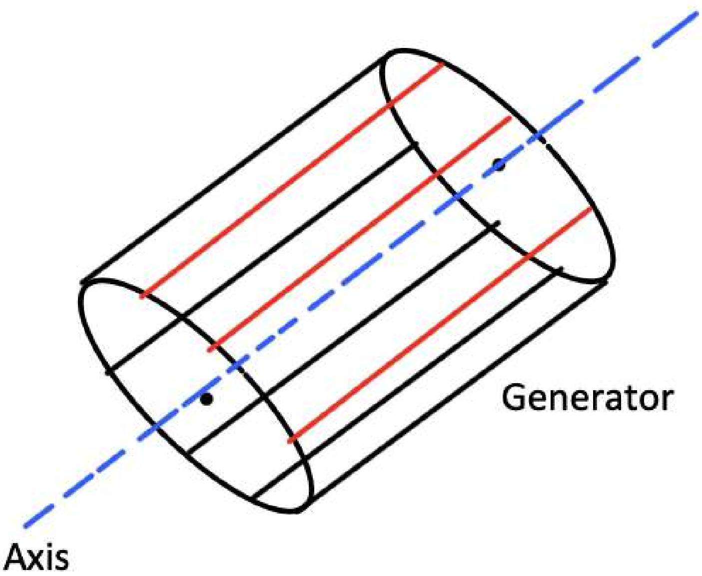
Suppose axis of the cone is on Z-axis.
The direction cosines of Z-axis are $0, 0, 1$.
Substituting $l = 0, m = 0, n = 1$ in equation (16), we get

$$z^2 = (x^2+y^2+z^2)\cos^2\alpha$$
$$\Rightarrow \quad z^2\sec^2\alpha = x^2+y^2+z^2$$
$$\Rightarrow z^2(\sec^2\alpha - 1) = x^2+y^2$$
$$\Rightarrow \quad x^2+y^2 = z^2\tan^2\alpha \tag{17}$$

Equation (17) represents the right circular cone having vertex at the origin and axis on Z-axis.

6.2 Cylinder

Definition 6.2 *A surface generated by a straight line which always remains parallel to the given fixed line and which intersects to the given curve is called* ***cylinder****. The stright line which generates the cylinder is called as* ***the generators*** *of the cylinder and the given curve is called as the* ***guiding curve****.*



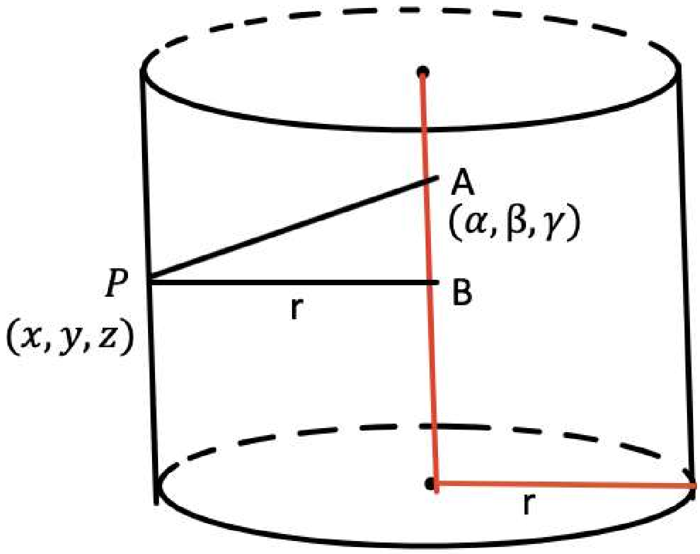
Equation of Cylinder

The equation of cylinder whose generator intersects the conic, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ z = 0$ and parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is given as

$$a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2fn(ny - mz) + cn^2 = 0$$

Right Circular Cylinder:

Definition 6.3 *A cylinder is called a right circular cylinder if its guiding curve is a circle and its generators are lines perpendicular to the plane containing the circle.*
The normal to the plane of a guiding circle passing through its center is called as ***the axis of the cylinder.***
The ***plane section perpendicular*** *to the axis of a* ***cylinder is a circle****. The radius of this circle is called as the* ***radius of the cylinder****.*



Right Cicular Cylinder

Equation of Right Circular Cylinder

Consider the right circular cylinder with radius r and axis given by

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

Refer the above figure of right circular cylinder.
Let $P(x, y, z)$ be any point on the right crcular cylinder and $A(\alpha, \beta, \gamma)$ be the point on the axis of cylinder.
Draw $PB \perp$ Axis and join PA.
$\therefore \quad PB = r$
The direction ratios of PA are $x - \alpha, y - \beta, z - \gamma$.

The direction ratios of AB are l, m, n. In right angled triangle PBA, we have

$$AB = (PA)\cos\angle PAB$$
$$AB = (PA)\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$
$$\text{But}\quad PA = \sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}$$
$$\Rightarrow AB = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}$$

$$\text{Now,}\quad (PA)^2 = (PB)^2 + (AB)^2$$
$$\Rightarrow (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 = r^2 + \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2}$$
$$(l^2+m^2+n^2)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2\right] = l(x-\alpha)+m(y-\beta)+n(z-\gamma)$$

This is the required equation.

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Practical 14: Equation of Cone

1. Find the equation of a cone with vertex at the point $(3, 1, 2)$ and guiding curve is $2x^2 + 3y^2 = 1, \ z = 0$.
2. Find the equation of the right circular cone with vertex at $(2, -1, 4)$, the line $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$ as the axis and semi-verticle angle $\cos^{-1}\left(\frac{4}{\sqrt{6}}\right)$.
3. Find the equation of the cone whose vertex $(1, 1, 2)$ and guiding curve is the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1, \ z = 0$.
4. Find the equation of cone passing through the coordinate axes and having the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x}{3} = \frac{y}{-1} = \frac{z}{-1}$ as generators.
5. Find the equation of a right circular cone whose vertex is at the origin and axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ with semivertical angle of 60^o
6. Show that the equation $7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$ represents a cone. Find the vertex,.
7. Show that the line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{3}$ is a generator of the cone $x^2 + y^2 + z^2 + 4xy - xz = 0$.
8. Show that the equation $4x^2 - y^2 - 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$ represents a cone with vertex at the point $(-1, -2, -3)$.
9. Determine the equation of right circular cone having vertex at $(2, 3, 1)$, axis parallel to the line $2x = -y = -2z$ and one of its generators having d.r.s $1, 1, 1$.

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Practical 15: Equation of Cylinder

1. Find the equation of a right circular cylinder of radius 2, whose axis passes through $(1, 2, 3)$ and has direction cosines proportional to $2,\ -3,\ 6$.

2. Find the equation of right circular cylinder of radius 3 whose axis lies along the line $\dfrac{x-1}{2} = \dfrac{y+3}{-1} = \dfrac{z-2}{5}$.

3. Obtain the equation of a right circular cylinder whose guiding curve is the circle $x^2 + y^2 + z^2 - 9 = 0,\ x - y + z - 3 = 0$.

4. Find the equation of right circular of radius 1, whose axis is the line of in tersection of two planes $x - 2y + 3 = 0,\ 2y - z - 1 = 0$.

5. Find the equation of the right circular cylinder whose axis is $\dfrac{x-2}{2} = \dfrac{y-1}{1} = \dfrac{z}{3}$ and which passes through the point $(0, 0, 1)$.

6. Find the equation of right circular cylinder of radius 2 whose axis lies is the line $\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z}{2}$.

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