Limit and Continuity of Multivariable functions

1.1 Functions of several variable

Definitions:

Real Valued functions of two independent variables:
Let $D = \{(x, y)|x, y \in R\} \subset R^2$. A real valued function f on D is a rule that assign a unique single real number $w = f(x, y)$ to each element in D.

Real Valued functions of three independent variables:
Let $D = \{(x, y, z)|x, y, z \in R\} \subset R^3$. A real valued function f on D is a rule that assign a unique single real number $w = f(x, y, z)$ to each element in D.

Real Valued functions of n independent variables:
Let $D = \{(x_1, x_2, ..., x_n)|x_i \in R\} \subset R^n$. A real valued function f on D is a rule that assign a unique single real number $w = f(x_1, x_2, ..., x_n)$ to each element in D.
Note: The set D is the domain of a function. The set $f(D) = \{w \in R|w = f(x_1, x_2, ...x_n)\}$ is the range of a function.
w is dependent variable and $x_1, x_2, ..., x_n$ are independent variables.

Examples:

1. Sketch the domain of $f(x, y) = \sqrt{y - x^2}$. What is the range of a function?
Solution: The domain D is the set of all pairs (x, y) in the plane for which $\sqrt{y - x^2}$ is real.$\therefore y - x^2 \geq 0 \therefore y \geq x^2$.
Therefore domain is set of al points lie above and on parabola $y = x^2$.
the range of a function is set of all non-negative numbers $w \geq 0$.

2 .What are the domain and range of the function $f(x, y) = \dfrac{1}{xy}$?
Solution: The domain D is set of all pairs in the plain for which $xy \neq 0$
The range of a function is the set $\{w \in R|w \in (-\infty, 0) \cup (0, \infty)\}$

3.What are the domain and range of the function $f(x, y) = \dfrac{xy}{x^2 - y^2}$?
Solution: The domain D is set of all pairs in the plain for which $x^2 - y^2 \neq 0$
that is all points except on the lines $y = x$ and $y = -x$
The range of a function is the entire set of real numbers i.e.$-\infty < w < \infty$

4.What are the domain and range of the function $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$?
The domain D is set of all pairs in the plain for which $x^2 + y^2 + z^2 \neq 0$
i.e. the domain is set of points $(x, y, z) \neq (0, 0, 0)$
The range of a function is the set of all positive real numbers i.e.$0 < w < \infty$

Definitions: Interior point: A point (x_0, y_0) in a region R in the xy plane is called an interior point of R if every disk centered at (x_0, y_0) of a positive radius lies entirely in R.

Boundary point: A point (x_0, y_0) in a region R in the xy plane is called an boundary point of R if every disk centered at (x_0, y_0) of a positive radius lie in R as well as points lie outside of R.

Open Region: A region is open if it consists entirely of interior points.

Closed Region: A region is open if it consists all its interior and boundary points.

Bounded Region: A region in the plane is bounded if it lies inside a disk of fixed radius.
Unbounded Region: A region in the plane is unbounded if it is not bounded.
For examples:
Bounded regions: Triangles, rectangles, circles, disks
Unbounded regions:lines, coordinates axes, quadrants, half planes

Example:
If $f(x, y) = \sqrt{y - x}$ then
1. find the boundary of domain of a function 2.determine whether domain is open region,, closed region or neither 3.decide whether domain is bounded or unbounded.
Solution: Let $f(x, y) = \sqrt{y - x}$
The domain of $f(x, y)$ is the set of all pairs for which $(y - x) \geq 0$
i.e. $y \geq x$ and the range is set of non-negative numbers $w \geq 0$
Boundary of domain of a function is $y = x$, the straight line passing through the origin. therefore domain is closed which is unbounded.

Graphs and level curves of functions of two variables
Definitions
Level curves: The set of all points in the plane where a function $f(x, y)$ has a constant value $'c'$ is called a level curve of f
i.e. $\{(x, y) \in D | f(x, y) = c\}$.
Graph of a function: The set of all points $\{x, y, f(x, y)\}$ in a space for (x, y) in domain of f is called the graph of the function f.
The graph of the function f is also called the surface $z = f(x, y)$.
Examples:
1.Plot the level curves for the function $f(x, y) = \frac{x+y}{x-y}, x \neq y$, if $c = 0, 1$.
Solution: For level curve $f(x, y) = c, c \in R$
$\frac{x+y}{x-y} = c \Rightarrow y = \frac{c-1}{c+1}x$
that is the level curves are the lines passing through origin with slope $\frac{c-1}{c+1}$
(1) For $c = 0$, slope is -1
$\therefore$ level curve is the line $y = -x$
(2) For $c = 1$, slope is 0
$\therefore$ level curve is $y = 0$ that is $x-$ axis.

2. Plot the graph and level curve of the function $f(x, y) = 100 - x^2 - y^2$ for $c = 0, 51, 75$.
Solution: Let $f(x, y) = 100 - x^2 - y^2$. The domain D of f is entire xy plane and range of f is the set of all real numbers less than or equal to 100.
For level curves: $f(x, y) = c, c \in R$
$100 - x^2 - y^2 = c$
$x^2 + y^2 = 100 - c$
Therefore, level curves are circles centered at origin with radius $\sqrt{100 - c}$
a.For $c = 0, x^2 + y^2 = 100$, level curve is the circle with center origin and radius 10
b.For $c = 51, x^2 + y^2 = 49$, level curve is the circle with center origin and radius 7
c.For $c = 75, x^2 + y^2 = 25$, level curve is the circle with center origin and radius 5

3.Sketch the level curves of $f(x, y) = -(x-1)^2 - y^2 + 1$ for $c = 1, 0, -3, -8$
Solution: Let $f(x, y) = -(x-1)^2 - y^2 + 1$
$-(x-1)^2 - y^2 + 1 = c \Rightarrow (x-1)^2 + y^2 = 1 - c$
level curves are circles centered at $(1, 0)$ and radius $\sqrt{1-c}$
i.e. level curves are circles centered at $(1, 0)$ and respective radii are $0, 1, 2, 3$
4. Find an equation of level curve of $f(x, y) = 16 - x^2 - y^2$ that passes through the point $(2\sqrt{2}, \sqrt{2})$.
Solution: Let level curves $f(x, y) = 16 - x^2 - y^2$ that passes through the point $(2\sqrt{2}, \sqrt{2})$.
$\therefore f(x, y) = 16 - (2\sqrt{2})^2 - (\sqrt{2})^2 = 6$
$16 - x^2 - y^2 = 6$
$x^2 + y^2 = 10$
Level curve is circle with centered at origin and radius is $\sqrt{10}$

1.3 Limit and Continuity **Limit of function of two variables**
A function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0), if there exist corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,
$|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$
or $0 < |x - x_0| < \delta$ and $0 < |y - y_0| < \delta$
Notation: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$

Properties of limits:
If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x, y) = M$ where $L, M \in R$ then
1. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x, y) \pm g(x, y) = L \pm M$
2. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x, y).g(x, y)] = L.M$
3. $\lim\limits_{(x,y)\to(x_0,y_0)} Kf(x, y) = KL, K \in R$
4. $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}, M \neq 0$

Note: If a function $f(x, y)$ has different limits along two different paths as $(x, y) \to (x_0, y_0)$ then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exists.

Continuous Functions of two variables:
Definition: A function $f(x, y)$ is said to be continuous at the point (x_0, y_0), if
1.$f(x, y)$ is defined at (x_0, y_0) i.e. $f(x_0, y_0)$ exists.
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

Definition: A function $f(x, y)$ is said to be continuous in a region R if it is continuous at every point of its domain.
Properties of continuous functions:
If $f(x, y)$ and $g(x, y)$ both are continuous at point (x_0, y_0) then
1.$f(x, y) \pm g(x, y)$ are continuous at point (x_0, y_0)
2.$Kf(x, y)$ is continuous at point (x_0, y_0)
3.$f(x, y)g(x, y)$ is continuous at point (x_0, y_0)

4.$\frac{f(x,y)}{g(x,y)}$ continuous at point $(x_0, y_0), g(x, y) \neq 0$
5.$|f(x, y)|$ is continuous at point (x_0, y_0)

Theorem: Let $f(x, y)$ be continuous function at point (x_0, y_0) then $f(x, y_0)$ is continuous at $x = x_0$ and $f(x_0, y)$ is continuous at $y = y_0$, where $f(x, y_0), f(x_0, y)$ being a continuous functions of one variable.
Proof: Since $f(x, y)$ be continuous function at point (x_0, y_0)
$\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$
For given $\epsilon > 0$, there exist $\delta > 0$ such that
$|x - x_0| < \delta$ and $|y - y_0| < \delta \Rightarrow |f(x, y) - f(x_0, y_0)| < \epsilon$
$|x - x_0| < \delta$ and $|y_0 - y_0| < \delta \Rightarrow |f(x, y_0) - f(x_0, y_0)| < \epsilon$
$|x - x_0| < \delta \Rightarrow |f(x, y_0) - f(x_0, y_0)| < \epsilon$
$f(x, y_0)$ is continuous at $x = x_0$
Similarly $f(x_0, y)$ is continuous at $y = y_0$

Examples:
1.Let $f(x, y) = \begin{cases} 2xyx^2 + y^2, (x, y) \neq (0, 0) \\ 0, (x, y) = (0, 0) \end{cases}$ Check the continuity of $f(x, y)$ at origin
Solution: Consider along the path $y = mx$
$\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{x\to 0} f(x, mx)$
$\Rightarrow \lim_{x\to 0} \frac{2mx^2}{x^2+mx^2}$
$\Rightarrow \frac{2m}{1+m^2}$
So limit depends on m therefore limit does not exists at $(0, 0)$
SO function is not continuous at $(0, 0)$
2.Let $f(x, y) = \frac{x+y}{2+cosx}$ and $\epsilon = 0.02$. Show that there exists $\delta > 0$ such that for all $(x, y), \sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - f(0, 0)| < \epsilon$
Solution: Let $f(x, y) = \frac{x+y}{2+cosx}$ and $\epsilon = 0.02$
Since $-1 \leq cosx \leq 1$
$(-1 + 2) \leq 2 + cosx \leq (2 + 1)$
$1 \leq 2 + cosx \leq 3$
$\frac{1}{3} \leq \frac{1}{2+cosx} \leq 1$
$\frac{|x+y|}{3} \leq |\frac{x+y}{2+cosx}| \leq |x + y| \leq |x| + |y|$
$|\frac{x+y}{2+cosx}| \leq |x| + |y|$
Consider $|f(x, y) - f(0, 0)| = |\frac{x+y}{2+cosx} - 0| = |\frac{x+y}{2+cosx}| \leq |x| + |y| \therefore |x| < \delta, |y| < \delta$ then $|f(x, y) - f(0, 0)| \leq |x| + |y| < 2\delta = \epsilon$
$\therefore$ there exist $\delta = \epsilon/2 = 0.02/2 = 0.01$
Such that $|x| < \delta, |y| < \delta \Rightarrow |f(x, y) - f(0, 0)| < \epsilon$

3. Let $f(x, y, z) = tan^2x + tan^2y + tan^2z$ and $\epsilon = 0.03$ show that there exists $\delta > 0$ such that for all $(x, y, z)\sqrt{x^2 + y^2 + z^2} < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| < \epsilon$
Solution: Let $f(x, y, z) = tan^2x + tan^2y + tan^2z$ and $\epsilon = 0.03$
Consider $|f(x, y, z) - f(0, 0, 0)| = |tan^2x + tan^2y + tan^2z| \leq |tan^2x| + |tan^2y| + |tan^3z| \leq tan^2x + tan^2y + tan^2z$

Now if $|x| < \delta, |y| < \delta, |z| < \delta$
$\Rightarrow |f(x,y,z) - f(0,0,0)| \leq tan^2x + tan^2y + tan^2z \leq\leq tan^2\delta + tan^2\delta + tan^2\delta = 3tan^2\delta = \epsilon$
$\therefore tan^2\delta = \epsilon/3 = 0.03/3 = 0.01$
$\therefore tan\delta = 0.1 \therefore$ there exists $\delta = tan^{0.1}$
such that $|x| < \delta, |y| < \delta, |z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| < \epsilon$

4.Show that $f(x,y) = \frac{x^2}{x^2-y}$ has no limit as $(x,y) \to (0,0)$ by considering different paths.
Solution: Along X-axis
$\lim_{(x,y)\to(0,0)} = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$
Along Y-axis
$\lim_{(x,y)\to(0,0)} = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{0-y} = 0$
Along $y = Kx^2, (K \neq 1)$
$\lim_{(x,y)\to(0,0)} = \lim_{x\to 0} f(x,Kx^2) = \lim_{x\to 0} \frac{x^2}{x^2-Kx^2} = \lim_{x\to 0} \frac{x^2}{x^2(1-K)} = \frac{1}{1-K}$ So different limits for different paths
Therefore limit does not exists.
5. At what points the function $f(x,y) = sin\frac{1}{xy}$ is continuous?
Solution:$f(x,y) = sin\frac{1}{xy}$ is continuous at all points except $x = 0$ or $y = 0$
6. At what points (x,y,z) in the space the function $f(x,y,z) = \frac{1}{|xy|+|z|}$ is continuous?
Solution: The function $f(x,y,z) = \frac{1}{|xy|+|z|}$ is continuous at all points (x,y,z) except $(0,y,0)$ or $(x,0,0)$
7. Define $f(0,0)$, so that $f(x,y) = ln(\frac{3x^2-x^2y^2+3y^2}{x^2+y^2})$ is continuous at origin.
Solution: Since $f(x,y)$ is continuous at origin.
$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$$\therefore f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} ln(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2})$$

Using polar coordinates $x = rcos\theta, y = rsin\theta$ so $r, \theta \to 0$ as $(x,y) \to (0,0)$

$$\therefore f(0,0) = \lim_{(r,\theta)\to(0,0)} ln\frac{(3r^2cos\theta - r^4cos^2\theta sin^2\theta + 3r^2sin^2\theta)}{r^2}$$

$$\therefore f(0,0) = \lim_{(r,\theta)\to(0,0)} ln(3 - r^2cos^2\theta sin^\theta) = ln3$$