Differential Geometry

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CHAPTER 1

GRAPHS AND LEVEL SETS

Definition. Given a function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n+1}$, it's level sets are the sets $f^{-1}(c)$ defined, for each real number c, by

$$f^{-1}(c) = \{(x_1, x_2, ..., x_{n+1}) \in U : f(x_1, x_2, ..., x_{n+1}) = c\}.$$

The number c is called the height of the level set, and $f^{-1}(c)$ is called level set at height c.

Note:
1. $f^{-1}(c)$ may contain one point if f is one-one.
2. $f^{-1}(c) = U$ if f is constant function.
3. $f^{-1}(c) = \phi$ if c is not the point in range set of f.

Example 1. Find the level set at height 0 where $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$.
Solution. Let $c = 0$.

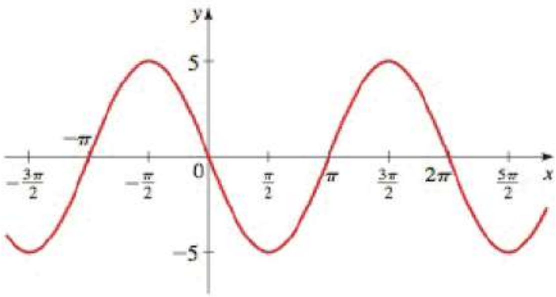
$$\begin{aligned} f^{-1}(0) &= \{x \in \mathbb{R} : f(x) = 0\} \\ &= \{x \in \mathbb{R} : \sin x = 0\} \\ &= \{\cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots\} \\ &= \{n\pi : n \in \mathbb{Z}\} \end{aligned}$$

which is level set at height 0.
Definition. The graph of function $f : U \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+2}$ defined by

$$\text{graph}(f) = \{(x_1, x_2, ..., x_{n+2}) \in \mathbb{R}^{n+2} : (x_1, x_2, ..., x_{n+1}) \in U \text{ and } x_{n+2} = f(x_1, x_2, ..., x_{n+1})\}$$

Example 2. Find the graph of function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$.
Solution.

$$\begin{aligned} \text{graph}(f) &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R} \text{ and } x_2 = f(x_1)\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R} \text{ and } x_2 = \sin x_1\} \\ &= \{(x_1, \sin x_1) \in \mathbb{R}^2\} \end{aligned}$$



Example 3. Find the level set $f^{-1}(c)$ for $n = 0, 1, 2$ at $c = 0, 1, 2, 3$ and $c = 4$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, x_2, ..., x_{n+1}) = x_1^2 + x_2^2 + ... + x_{n+1}^2$.
Solution. For $n = 0$, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x_1) = x_1^2$.
For $c = 0$

$$\begin{aligned} f^{-1}(0) &= \{x \in \mathbb{R} : f(x) = 0\} \\ &= \{x \in \mathbb{R} : x^2 = 0\} \\ &= \{x \in \mathbb{R} : x = 0\} \\ &= \{0\} \end{aligned}$$

For $c = 1$

$$\begin{aligned} f^{-1}(1) &= \{x \in \mathbb{R} : f(x) = 1\} \\ &= \{x \in \mathbb{R} : x^2 = 1\} \\ &= \{x \in \mathbb{R} : x = -1, 1\} \\ &= \{-1, 1\} \end{aligned}$$

For $c = 2$

$$\begin{aligned} f^{-1}(2) &= \{x \in \mathbb{R} : f(x) = 2\} \\ &= \{x \in \mathbb{R} : x^2 = 2\} \\ &= \left\{x \in \mathbb{R} : x = -\sqrt{2}, \sqrt{2}\right\} \\ &= \left\{-\sqrt{2}, \sqrt{2}\right\} \end{aligned}$$

For $c = 3$

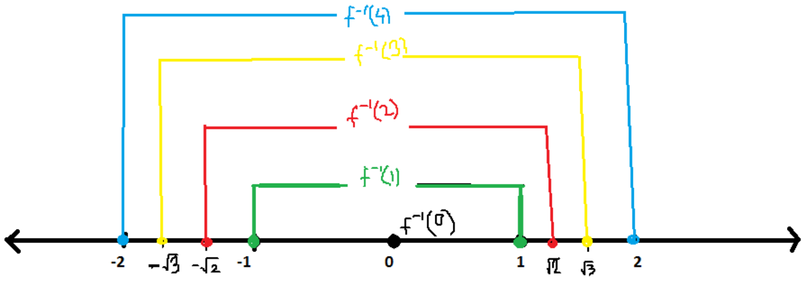
$$\begin{aligned} f^{-1}(3) &= \{x \in \mathbb{R} : f(x) = 3\} \\ &= \{x \in \mathbb{R} : x^2 = 3\} \\ &= \left\{x \in \mathbb{R} : x = -\sqrt{3}, \sqrt{3}\right\} \\ &= \left\{-\sqrt{3}, \sqrt{3}\right\} \end{aligned}$$

For $c = 4$

$$\begin{aligned} f^{-1}(4) &= \{x \in \mathbb{R} : f(x) = 4\} \\ &= \{x \in \mathbb{R} : x^2 = 4\} \\ &= \{x \in \mathbb{R} : x = -2, 2\} \\ &= \{-2, 2\} \end{aligned}$$

For $n = 1$, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2 + x_2^2$.
For $c = 0$,

$$\begin{aligned} f^{-1}(0) &= \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = 0\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 0\} \\ &= \{(0, 0)\} \end{aligned}$$



Level set (n = 0)

For $c = 1$,

$$\begin{aligned} f^{-1}(1) &= \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = 1\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \end{aligned}$$

For $c = 2$,

$$\begin{aligned} f^{-1}(2) &= \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = 2\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 2\} \end{aligned}$$

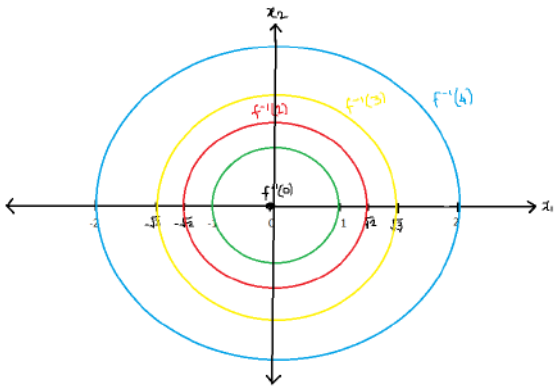
For $c = 3$,

$$\begin{aligned} f^{-1}(3) &= \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = 3\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 3\} \end{aligned}$$

For $c = 4$,

$$\begin{aligned} f^{-1}(4) &= \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = 4\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 4\} \end{aligned}$$

For $n = 2$, $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$.



Level set (n = 1)

For $c = 0$,

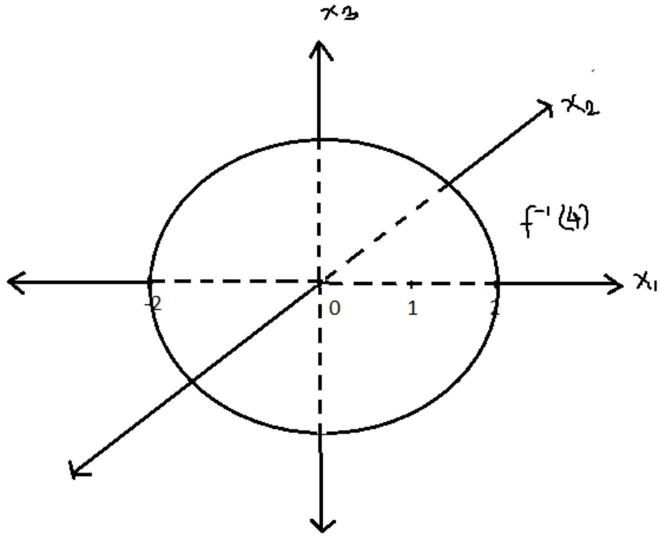
$$\begin{aligned} f^{-1}(0) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(x_1, x_2, x_3) = 0\} \\ &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 0\} \\ &= \{(0, 0, 0)\} \end{aligned}$$

For $c = 1$,

$$\begin{aligned} f^{-1}(1) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(x_1, x_2, x_3) = 1\} \\ &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \end{aligned}$$

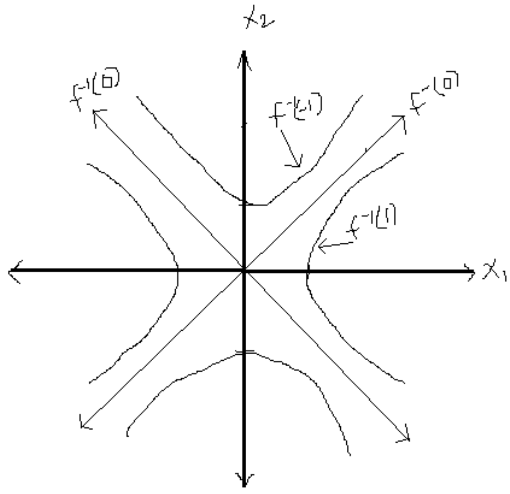
For $c = 4$,

$$\begin{aligned} f^{-1}(4) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(x_1, x_2, x_3) = 4\} \\ &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 4\} \end{aligned}$$



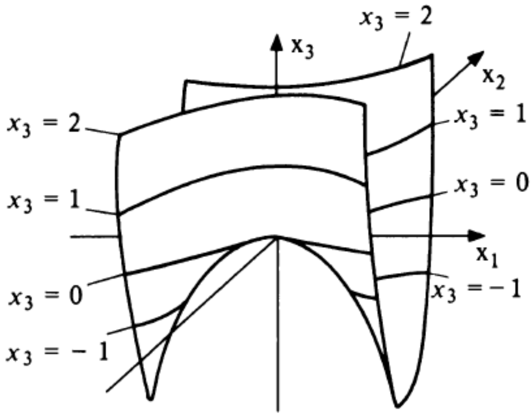
Level set (n = 2)

Example 4. Find the typical level curves and the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = -x_1^2 + x_2^2$.
Solution. Level set:



Level set at height c = 0, -1, 1

Graph:



Graph of $f(x_1, x_2) = -x_1^2 + x_2^2$

Example 5. Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.
PROOF. Let $f : \mathbb{R}^n \to \mathbb{R}$. Then

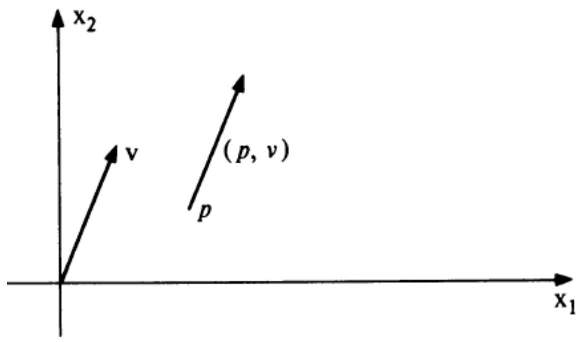
$$\text{graph}(f) = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, x_2, ..., x_n) \in \mathbb{R}^n \text{ and } x_{n+1} = f(x_1, x_2, ..., x_n)\}$$

Now we define $F : \mathbb{R}^{n+1} \to \mathbb{R}$ as $F(x_1, x_2, ..., x_{n+1}) = f(x_1, x_2, ..., x_n) - x_{n+1}$.
Then

$$\begin{aligned} F^{-1}(0) &= \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : F(x_1, x_2, ..., x_{n+1}) = 0\} \\ &= \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : f(x_1, x_2, ..., x_n) - x_{n+1} = 0\} \\ &= \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = f(x_1, x_2, ..., x_n)\} \\ &= \text{graph}(f) \end{aligned}$$

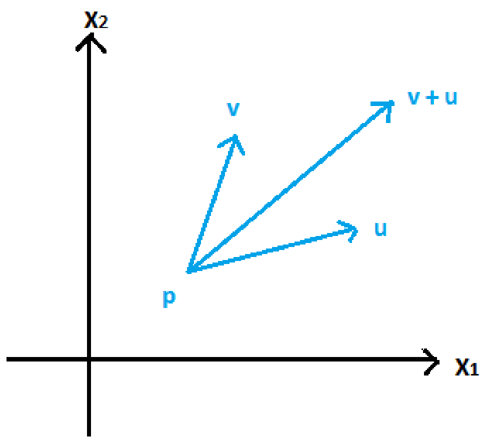
CHAPTER 2
VECTOR FIELDS

Definition. A vector at a point $p \in \mathbb{R}^{n+1}$ is a pair (p, v) where $v \in \mathbb{R}^{n+1}$.
Geometrically, think of v as the vector v translated so that its tail is at p rather than at origin.



A vector at *p*.

The vectors at p form a vector field $\mathbb{R}^{n+1}_p$ of dimension $n+1$, with addition defined by $(p, v) + (p, w) = (p, v + w)$ and scalar multiplication by $c(p, v) = (p, cv)$.



Addition of vectors at p

If $\{v_1, v_2, ..., v_{n+1}\}$ is any basis for $\mathbb{R}^{n+1}$ then $\{(p, v_1), (p, v_2), ..., (p, v_{n+1})\}$ forms a basis for $\mathbb{R}^{n+1}_p$.
DEFINITIONS:

Dot product. Given two vectors (p,v) and (p,w) at p, then their dot product is defined using standard dot product on $\mathbb{R}^{n+1}$, by $(p,v)\cdot(p,w) = v\cdot w$.
Cross product. Given two vectors (p,v) and $(p,w) \in \mathbb{R}^3_p$, where $p \in \mathbb{R}^3$, then their cross product is also defined, using the standard cross product on $\mathbb{R}^3$, by $(p,v)\times(p,w) = (p, v\times w)$.
Length of vector. The length of a vector $v = (p,v)$ at p is

$$\begin{aligned} \|v\| &= (v\cdot v)^{1/2} \\ &= ((p,v)\cdot(p,v))^{1/2}. \end{aligned}$$

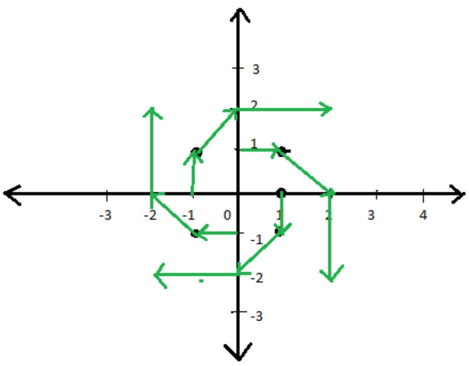
Angle between two vectors. The angle between two vectors $v = (p,v)$ and $w = (p,w)$ is

$$\cos\theta = \frac{v\cdot w}{\|v\|\ \|w\|}, \text{ where } 0 \le \theta < \pi.$$

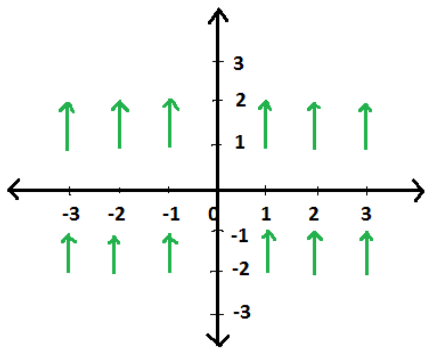
Vector field. A vector field X on $U \subset \mathbb{R}^{n+1}$ is a function which assigns to each vector of U a vector at that point, Thus

$$X(p) = (p, X(p)).$$

Example 1. The sketch of some vector fields $X : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $X(x_1,x_2) = (x_2, -x_1)$ and $X(x_1,x_2) = (0,1)$ are given below:



$X(x_1, x_2) = (x_2, -x_1)$



$X(p) = (0, 1)$

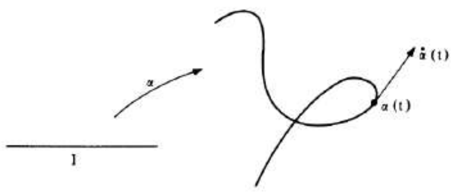
Open set. A set $U \subset \mathbb{R}^{n+1}$ is open if for each point $p \in U$ there is an $\epsilon > 0$ such that $q \in U$ whenever $\|q-p\| < \epsilon$.
Smooth function. A function $f : U \to \mathbb{R}$, where U is open subset of $\mathbb{R}^{n+1}$ is called smooth function if all it's partial derivatives of all orders are exists and continuous.
A function $f : U \to \mathbb{R}^k$ where U is open subset of $\mathbb{R}^{n+1}$ is called smooth function if each component function $f_i : U \to \mathbb{R}(f(p) = (f_1(p), f_2(p), ..., f_{n+1}(p))$ for $p \in U)$ is smooth.
A vector field X on U is smooth if the associated function $X : U \to \mathbb{R}^{n+1}$ is smooth.
Gradient of a function. Associated with each smooth function $f : U \to \mathbb{R}(U$ open in $\mathbb{R}^{n+1})$ is a smooth vector field on U called gradient of f defined by

$$(\nabla f)(p) = \left(p, \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), ..., \frac{\partial f}{\partial x_{n+1}}(p)\right).$$

Parametric curve. A parametric curve in $\mathbb{R}^{n+1}$ is a smooth function $\alpha : I \to \mathbb{R}^{n+1}$, where I is some open interval in $\mathbb{R}$. It has the form $\alpha(t) = (x_1(t), x_2(t), ..., x_{n+1}(t))$ where each x_i is a smooth real valued function on I.
Velocity vector. The velocity vector at time $t \in I$ of parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is the vector at $\alpha(t)$ defined by

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), ..., \frac{dx_{n+1}}{dt}(t)\right).$$

This vector is tangent to the curve α at $\alpha(t)$.



Velocity vector of a parametrized curve in $\mathbb{R}^2$.

Integral curve. A parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is said to be integral curve of the vector field X on the open set U in $\mathbb{R}^{n+1}$ if $\alpha(t) \in U$ and $\dot{\alpha}(t) = X(\alpha(t))$ for all $t \in I$.

An integral curve of a vector field.

Theorem. *Let X be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of X such that*
(i) $\alpha(0) = p$.
(ii) If $\beta : \tilde{I} \to U$ is any another integral curve of X with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $\tilde{I}$.
PROOF. Since X is a smooth vector field on U hence it has the form

$$X(p) = (p, X_1(p), X_2(p), ..., X_{n+1}(p))$$

where, each $X_i : U \to \mathbb{R}$ is smooth functions on U. A parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ has the form .

$$\alpha(t) = (x_1(t), x_2(t), ..., x_{n+1}(t))$$

where, each $x_i : I \to \mathbb{R}$ is smooth function on I. The velocity of α is

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), ..., \frac{dx_{n+1}}{dt}(t)\right)$$

Suppose $\alpha : I \to$ be an integral curve of a vector field X

$$\Longrightarrow \dot{\alpha(t)} = X(\alpha(t))$$

$$\Longrightarrow \left(\alpha(t), \frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), ..., \frac{dx_{n+1}}{dt}(t)\right) = (\alpha(t), X_1(\alpha(t)), X_2(\alpha(t)), ..., X_{n+1}(\alpha(t)))$$

Equating components from both sides we get,

$$\begin{aligned} \frac{dx_1}{dt}(t) &= X_1(\alpha(t)) \\ \frac{dx_2}{dt}(t) &= X_2(\alpha(t)) \\ &. \\ &. \\ &. \\ \frac{dx_{n+1}}{dt}(t) &= X_{n+1}(\alpha(t)) \end{aligned}$$

This is the system of $n+1$ first order ordinary differential equations in $n+1$ unknowns. Therefore, by existence theorem for solutions of such equations there exists and open interval I containing 0 and set $x_i : I_1 \to \mathbb{R}$ of smooth functions satisfying this system subject to initial conditions $x_i(0) = p$ for $i \in \{1, 2, ..., n+1\}$, where $p = (p_1, p_2, ..., p_{n+1})$. Setting $\beta_1(t) = (x_1(t), x_2(t), ..., x_{n+1}(t))$ for this choice of functions we get an integral curve $\beta_1 : I_1 \to U$ of X with $\beta_1(0) = p$.
Also, by uniqueness theorem for the solutions of first order ordinary differential equations, if $\tilde{x}_i : I_2 \to \mathbb{R}$ is another set of functions satisfying the given system together with the initial conditions $\tilde{x}_i(0) = p_i$ then $\tilde{x}_i(t) = x_i(t)$ for all $t \in I_1 \cap I_2$.
In other words, if $\beta_2 : I_2 \to U$ is another integral curve of X with $\beta(0) = p$ then $\beta_1(t) = \beta_2(t)$ for all $t \in I_1 \cap I_2$.
It follows from this that there is unique maximal integral curve α of X with $\alpha(0) = p$ and if $\beta : \tilde{I} \to U$ is any another integral curve of X with $\beta(0) = p$ then β is simply restriction of α to the smaller interval $\tilde{I}$. ■

Example. Find the integral curve of vector field X, where $X(x_1, x_2) = (-x_2, x_1)$ through the point (a, b).

Solution. Suppose a parametric curve $\alpha(t) = (x_1(t), x_2(t))$ is an integral curve of X.

$$\begin{aligned} \Longrightarrow \dot{\alpha}(t) &= X(\alpha(t)) \\ \Longrightarrow \left(\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t)\right) &= X(x_1(t), x_2(t)) \\ \Longrightarrow \left(\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t)\right) &= (-x_2(t), x_1(t)) \end{aligned}$$

Equating components from both sides we get,

$$\frac{dx_1}{dt}(t) = -x_2(t) \quad (1)$$

$$\frac{dx_2}{dt}(t) = x_1(t) \quad (2)$$

Differentiating both the equations with respect to t we get,

$$\frac{d^2x_1}{dt^2} = -\frac{dx_2}{dt} \quad (3)$$

$$\frac{d^2x_2}{dt^2} = \frac{dx_1}{dt} \quad (4)$$

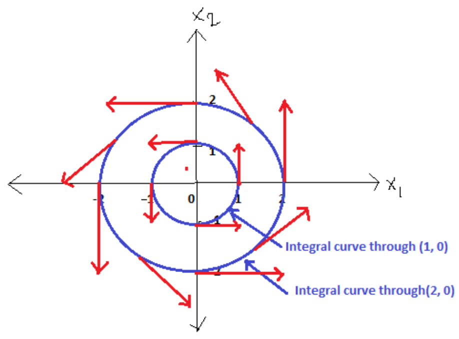
From equation (1) and (3) we get,

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -x_1 \\ \implies \frac{d^2x_1}{dt^2} + x_1 &= 0 \quad (5) \end{aligned}$$

The auxiliary equation of differential equation (5) is,

$$\begin{aligned} m^2 + 1 &= 0 \\ \implies m &= \pm i \end{aligned}$$

Therefore, the solution of differential equation (5) is $x_1(t) = c_1 \cos t + c_2 \sin t$.
$x_2(t) = -\dfrac{dx_1}{dt} = -(-c_1 \sin t + c_2 \cos t)$.
$\implies x_2(t) = c_1 \sin t - c_2 \cos t$.
Since the given integral curve passes through the point (a, b).
$\implies x_1(0) = a$ and $x_2(0) = b$.
Using these initial conditions we get $c_1 = a$ and $c_2 = -b$.
Therefore, the required integral curve is $\alpha(x_1, x_2) = (a \cos t - b \sin t, a \sin t + b \cos t)$.



Integral curves of the vector field X(x1, x2) = (-x2, x1)

Definition. The divergence of a smooth vector field X on $U \subset R^{n+1}$, $X(p) = (p, X_1(p), X_2(p), ..., X_{n+1}(p$
for $p \in U$, is the function $\text{div}X : U \to R$ defined by $\text{div}X = \sum_{i=1}^{n+1} \frac{\partial X_i}{\partial x_i}$.

Example 1. Find divergence of a vector field defined by $X(p) = (1, 0)$.

Solution. $\text{div}X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 0$.

Example 2. Find the divergence of a vector field defined by $X(p) = p$.

Solution. We have $X(x_1, x_2) = (x_1, x_2)$.

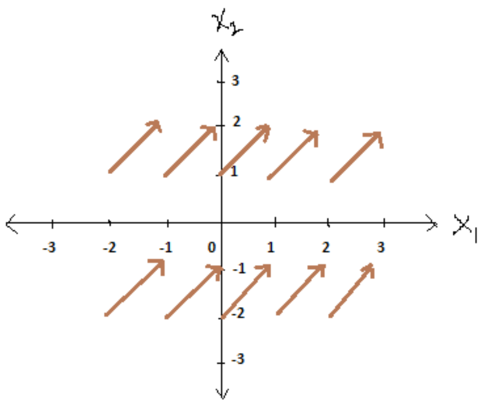
Therefore, $\text{div}X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 1 + 1 = 2$.

Example 3. Find and sketch the gradient field of each of the following functions:

(a) $f(x_1, x_2) = x_1 + x_2$

Solution. $\nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (1, 1)$.

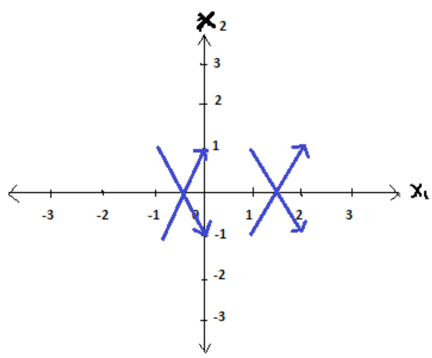
The sketch of this gradient vector field is given below:



(b) $f(x_1, x_2) = x_1 - x_2^2$

Solution. $\nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (1, -2x_2)$.

The sketch of this gradient vector field is given below:



Example 4. Explain why an integral curve of a vector field cannot cross itself as does the parametrized curve.
Solution. Let X be a smooth vector field on $U \subset R^{n+1}$.
On contrary assume that the integral curve crosses itself.
$\Longrightarrow \alpha(t_1) = \alpha(t_2)$, for some $t_1, t_2 \in I$ and $t_1 \neq t_2$.
Since α is integral curve of a vector field X.
$\Longrightarrow \dot{\alpha}(t_1) = X(\alpha(t_1))$ and $\dot{\alpha}(t_2) = X(\alpha(t_2))$.
But then, $\dot{\alpha}(t_1) = X(\alpha(t_1)) = X(\alpha(t_2)) = \dot{\alpha}(t_2)$.
$\Longrightarrow \dot{\alpha}(t_1) = \dot{\alpha}(t_2)$.
Which is not possible. Therefore, Integral curve of a smooth vector field X does not cross itself.
Definition. A smooth vector field X on an open set $U \subset R^{n+1}$ is said to be complete if for each $p \in U$ the maximal integral curve of X through p has domain equal to $\mathbb{R}$.
Example 5. Determine which of the following vector fields are complete.
(a) $X(x_1, x_2) = (x_1, x_2, 1, 0), U = \mathbb{R}^2$.
Solution. Here vector field X is defined on an open set $U = \mathbb{R}^2$. We define maximal integral curve $\alpha(t) = (x_1(t), x_2(t))$ of given vector field passing through (a, b).

$$\begin{aligned} \Longrightarrow \dot{\alpha}(t) &= X(\alpha(t)) \\ \Longrightarrow \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right) &= (1, 0) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{dx_1}{dt} &= 1 \\ \frac{dx_2}{dt} &= 0 \end{aligned}$$

$$\begin{aligned} x_1(t) &= t + c_1 \\ x_2(t) &= c_2 \end{aligned}$$

Since the integral curve passes through (a, b).
Therefore, $x_1(0) = a$ and $x_2(0) = b$.
$\Longrightarrow c_1 = a$ and $c_2 = b$.
Therefore, the integral curve is $\alpha(t) = (t + a, b)$.
For any $t \in \mathbb{R}, \alpha(t) = (a + t, b) \in \mathbb{R}^2$.
Therefore, domain of maximal integral curve is $\mathbb{R}$.
Therefore, the given vector field is complete.
(b) $X(x_1, x_2) = (x_1, x_2, 1, 0), U = \mathbb{R}^2 - \{(0, 0)\}$.
Solution. We have maximal integral curve of given vector field is $\alpha(t) = (t + a, b)$, where $U = \mathbb{R}^2 - \{(0, 0)\}$.
Now, for $t = 0, \alpha(0) = (a, b)$.
At the point $(a, b) = (0, 0), \alpha(0) = (0, 0)$.
But $\alpha(0) = (0, 0) \notin U \Longrightarrow 0 \notin \mathbb{R}$.

$\implies$ Domain of α not equal to $\mathbb{R}$.
Therefore, the given vector field is not complete.
Example 6. Show that $\alpha(t) = \left(\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t\right)$ is parametrization of the intersection of circular cylinder of radius $\frac{1}{2}$ and axis the z-axis with the sphere of radius 1 and centre is $\left(-\frac{1}{2}, 0, 0\right)$.
Solution. The equation of sphere whose centre at $\left(-\frac{1}{2}, 0, 0\right)$ and radius 1 is,

$$\left(x + \frac{1}{2}\right)^2 + y^2 + z^2 = 1 \qquad (1)$$

The equation of circular cylinder of radius 1/2 is,

$$x^2 + y^2 = \frac{1}{4} \qquad (2)$$

From equation (1) we have $z^2 \leq 1$.
$\implies -1 \leq z \leq 1$.
Substitute $z = \sin t$.
Subtracting equation (2) from equation (1) we get,

$$\begin{aligned}
\left(x + \frac{1}{2}\right)^2 - x^2 + z^2 &= 1 - \frac{1}{4} \\
x + \frac{1}{4} + z^2 &= \frac{3}{4} \\
x + z^2 &= \frac{1}{2} \\
x &= \frac{1}{2} - z^2 \\
x &= \frac{1}{2} - \sin^2 t \qquad \because z = \sin t \\
x &= \frac{1}{2} - \left(1 - \cos^2 t\right) \\
x &= \cos^2 t - \frac{1}{2}
\end{aligned}$$

Substituting this value of x in equation (2) we get,

$$\begin{aligned}\left(\cos^2 t - \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4}\\ \cos^4 t - \cos^2 t + \frac{1}{4} + y^2 &= \frac{1}{4}\\ y^2 &= \cos^2 t - \cos^4 t\\ y^2 &= \cos^2 t(1-\cos^2 t)\\ y^2 &= \cos^2 t \sin^2 t\\ y &= \cos t \sin t\end{aligned}$$

Therefore, $\alpha(t) = \left(\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t\right)$ is parametrization of the intersection circular cylinder of radius $\frac{1}{2}$ and axis the z-axis with the sphere of radius 1 and centre is $\left(-\frac{1}{2}, 0, 0\right)$.

Example 6. Explain why an integral curve of a vector field does not cross itself as does the parametrized curve.

Solution. Let X be a smooth vector field on U is subset of $\mathbb{R}^{n+1}$ and $\alpha : I \to U$ be an integral curve of X.

On contrary assume that the integral curve α cross with itself.

That is, there exists $t_1 \neq t_2 \in I$ such that $\alpha(t_1) = \alpha(t_2)$.

Since α is integral curve of X

$$\begin{aligned}\implies \dot{\alpha}(t_1) &= X(\alpha(t_1)) \text{ and}\\ \dot{\alpha}(t_2) &= X(\alpha(t_2))\end{aligned}$$

But then

$$\begin{aligned}\dot{\alpha}(t_1) = X(\alpha(t_1)) &= X(\alpha(t_2)) = \dot{\alpha}(t_2)\\ \implies \dot{\alpha}(t_1) &= \dot{\alpha}(t_2)\end{aligned}$$

Which is not possible.

Therefore, integral curve does not cross itself.

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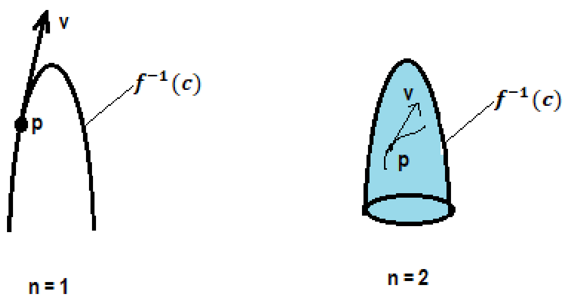
CHAPTER 3

THE TANGENT SPACES AND SURFACE

Definition. Let $f : U \to \mathbb{R}$ be a smooth function, where $U \subset \mathbb{R}^{n+1}$ is an open set, let $c \in \mathbb{R}$ be such that $f^{-1}(c)$ is non-empty, and let $p \in f^{-1}(c)$. A vector at p is said to be tangent to the level set $f^{-1}(c)$ if it is velocity vector of a parametrized curve whose image is contained in $f^{-1}(c)$ (see figure below).



Tangent vectors to level set

Therefore, the tangent vector to $f^{-1}(c)$ is of the form $\dot{\alpha}(t_0)$ for some parametrized curve $\alpha : U \to \mathbb{R}^{n+1}$ with $\alpha(t_0) = p$ and $\text{Im}(\alpha) \subset f^{-1}(c)$.
Lemma. *The gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p.*
PROOF. Each vector tangent to $f^{-1}(c)$ is of the form $\dot{\alpha}(t_0)$ for some parametrized curve $\alpha : U \to \mathbb{R}^{n+1}$ with $\alpha(t_0) = p$ and $\text{Im}(\alpha) \subset f^{-1}(c)$.
But $\text{Im}(\alpha) \subset f^{-1}(c) \implies f(\alpha(t)) = c, \forall t \in I$.
By chain rule of derivative we have,

$$\begin{aligned} \frac{d}{dt}(f \circ \alpha)(t_0) &= \nabla f(\alpha(t_0)) \cdot \dot{\alpha}(t_0) \\ \implies \frac{d}{dt}(c) &= \nabla f(p) \cdot \dot{\alpha}(t_0) \\ \implies \nabla f(p) \cdot \dot{\alpha}(t_0) &= 0 \end{aligned}$$

$\therefore$ Gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p. ■
Remark. If $\nabla f(p) = 0$ then lemma says nothing. But if $\nabla f(p) \neq 0$, it says that the set of all vectors tangent to $f^{-1}(c)$ at p is contained in the $n-$dimensional vector subspace $[\nabla f(p)]^\perp$ of $\mathbb{R}^{n+1}_p$ consisting of all vectors orthogonal to $\nabla f(p)$.
Definition. A point $p \in \mathbb{R}^{n+1}$ such that $\nabla f(p) \neq 0$ is called regular point of f.
Theorem. *Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let p be a regular point of f, and let $c = f(p)$. Then the set of all vectors tangent to $f^{-1}(c)$ at p is equal to*

$[\nabla f(p)]^{\perp}$.

PROOF. From the previous lemma we have that, every vector tangent to $f^{-1}(c)$ at p is contained in $[\nabla f(p)]^{\perp}$. Thus it suffices to show that, if $v = (p, v) \in [\nabla f(p)]^{\perp}$, then $v = \dot{\alpha}(0)$ for some parametrized curve α with $\text{Im}(\alpha) \subset f^{-1}(c)$. To construct α, consider the constant vector field X on U defined by $X(q) = (q, v)$. From X we can construct another vector field Y by subtracting from X the components of X along α.

$$Y(q) = X(q) - \frac{X(q) \cdot \nabla f(q)}{\|\nabla f(q)\|^2} \nabla f(q)$$

The vector field Y has domain U where $\nabla f \neq 0$. Since p is regular point of f, hence it is in domain of Y. Moreover, since $X(p) = v \in [\nabla f(p)]^{\perp}$. Therefore, $X(p) = Y(p)$. Here we have obtained smooth vector field Y such that $Y(q) \perp \nabla f(q), \forall q \in \text{domain}(Y)$, and $Y(p) = v$.

Now let α be an integral curve of Y through p.

Then $\alpha(0) = p, \dot{\alpha}(0) = Y(\alpha(0)) = Y(p) = X(p) = v$ and

$$\begin{aligned} \frac{d}{dt} f(\alpha(t)) &= \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) \\ &= \nabla f(\alpha(t)) \cdot Y(\alpha(t)) \\ &= 0 \end{aligned}$$

for all $t \in \text{domain}(\alpha)$, so that $f(\alpha(t)) = \text{constant}$. Since $f(\alpha(0)) = f(p) = c$, this means that $\text{Image}(\alpha) \subset f^{-1}(c)$. ■

Definition. A surface of dimension n or $n-$surface, in $\mathbb{R}^{n+1}$ is a non-empty subset S of $\mathbb{R}^{n+1}$ of the form $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$, U is open in $\mathbb{R}^{n+1}$, is a smooth function with the property that $\nabla f(p) \neq 0, \forall p \in S$.

A 1-surface in $\mathbb{R}^2$ is called a plane curve. A 2-surface in $\mathbb{R}^3$ is called simply a surface. An $n-$surface in $\mathbb{R}^{n+1}$ is called a hypersurface.

By theorem in previous chapter each $n-$surface S in $\mathbb{R}^{n+1}$, at each point $p \in S$ has tangent space which is $n-$dimensional vector surface of the space $\mathbb{R}^{n+1}_p$. This tangent space is denoted by S_p.

If f is smooth function and $S = f^{-1}(c)$ for some $c \in \mathbb{R}$ and $\nabla f(p) \neq 0, \forall p \in S$, then S_p may also be described as $[\nabla f(p)]^{\perp}$.

Example 1. Show that the unit $n-$sphere is a $n-$surface in $\mathbb{R}^{n+1}$.

Solution. The unit $n-$sphere $x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1$ represent the set $S = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$.

Here $S = f^{-1}(1)$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(x_1, x_2, ..., x_{n+1}) = x_1^2 + x_2^2 + ... + x_{n+1}^2$$

Since $(1, 0, 0, ..., 0) \in S$, hence S is non-empty subset of $\mathbb{R}^{n+1}$. Now for any $p = (x_1, x_2, ..., x_{n+1}) \in S$

$$\nabla f(p) = (p, 2x_1, 2x_2, ..., 2x_{n+1})$$

Therefore, $\nabla f(p) = 0 \iff (2x_1, 2x_2, ..., 2x_{n+1}) = 0 \iff p = (0, 0, ..., 0)$.
But $(0, 0, ..., 0) \notin S \implies \nabla f(p) \neq 0, \quad \forall p \in S$.
Therefore, S is n-surface in $\mathbb{R}^{n+1}$.
For $n = 1$, S is unit circle which is 1-surface in $\mathbb{R}^2$, for $n = 2$, S is sphere which is 2-surface in $\mathbb{R}^3$.

Example 2. Show that for $0 \neq (a_1, a_2, ..., a_{n+1}) \in \mathbb{R}^{n+1}$ and $b \in \mathbb{R}$, the n-plane $a_1x_1 + a_2x_2 + ... + a_{n+1}x_{n+1} = b$ is a n-surface in $\mathbb{R}^{n+1}$.

Solution. The n-plane $a_1x_1 + a_2x_2 + ... + a_{n+1}x_{n+1} = b$ represent the set $S = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : a_1x_1 + a_2x_2 + ... + a_{n+1}x_{n+1} = b\}$.
Here, $S = f^{-1}(b)$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$f(x_1, x_2, ..., x_{n+1}) = a_1x_1 + a_2x_2 + ... + a_{n+1}x_{n+1}$$

Since, $(b/a_1, 0, ..., 0) \in S$ hence S is non-empty subset of $\mathbb{R}^{n+1}$. Now for any $p = (x_1, x_2, ..., x_{n+1}) \in S$.

$$\nabla f(p) \quad = \quad (p, a_1, a_2, ..., a_{n+1})$$

Therefore, $\nabla f(p) = 0 \iff (a_1, a_2, ..., a_{n+1}) = 0$.
But $(a_1, a_2, ..., a_{n+1}) \neq 0 \implies \nabla f(p) \neq 0, \quad \forall p \in S$.
Therefore, S is n-surface in $\mathbb{R}^{n+1}$.
1-plane is usually called line in $\mathbb{R}^2$, 2-plane is called simply plane in $\mathbb{R}^3$ and an n-plane for $n > 2$ is called a hyperplane in $\mathbb{R}^{n+1}$. Two different values of b with the same value of $(a_1, a_2, ..., a_{n+1})$ defines parallel n-planes.

Example 3. Show that the graph of function $f : U \to \mathbb{R}$, where U is open subset of R^{n+1} is an n-surface in $\mathbb{R}^{n+1}$.

Solution. Let $S = \text{graph} f = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = f(x_1, x_2, ..., x_n)\}$.
Since graph$(f) = F^{-1}(0)$, for some $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$F(x_1, x_2, ..., x_n) = x_{n+1} - f(x_1, x_2, ..., x_n)$$

Now for any $p = (x_1, x_2, ..., x_{n+1}) \in S$,

$$\nabla F(p) \quad = \quad \left(p, -\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, ..., -\frac{\partial f}{\partial x_n}, 1\right)$$

$\therefore \nabla F(p) \neq 0, \forall p \in S$.
Therefore, gr(f) is a n-surface in $\mathbb{R}^{n+1}$.

Theorem. *Let S be an n-surface in $\mathbb{R}^{n+1}, S = f^{-1}(c)$ where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S; i.e. either $g(q) \leq g(p)$ for all $q \in S$ or $g(q) \geq g(p)$ for all $q \in S$. Then there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.*(The number λ is called a Lagrange multiplier.)

PROOF. Let S be an n-surface in $\mathbb{R}^{n+1}$.
Therefore, $S = f^{-1}(c)$, for some smooth function $f : U \to \mathbb{R}$ such that $\nabla f(p) \neq 0, \quad \forall p \in S$.
The tangent space to S at p is $S_p = [\nabla f(p)]^{\perp}$. Hence $S_p^{\perp} = [\nabla f(p)]$ is one dimensional

vector subspace of $\mathbb{R}^{n+1}_p$ spanned by $\nabla f(p)$.
To prove: $\nabla g(p) = \lambda \nabla f(p)$.
That is, to prove $\nabla g(p) \in S_p^{\perp}$.
That is, to prove $\nabla g(p) \cdot v = 0, \quad \forall v \in S_p$.
But each $v \in S_p$ is of the form $v = \dot{\alpha}(t_0)$ for some parametrized curve $\alpha : I \to S$ and $t_0 \in I$ with $\alpha(t_0) = p$. Since $p = \alpha(t_0)$ is extreme point of g on S.
$\Longrightarrow g(q) \leq g(p) \quad \forall q \in S$ or $g(q) \geq g(p) \quad \forall q \in S$.
$\Longrightarrow g(q) \leq g(\alpha(t_0)) \quad \forall q \in S$ or $g(q) \geq g(\alpha(t_0)) \quad \forall q \in S$.
Since, $\alpha : I \to S \Longrightarrow \alpha(t) \in S, \quad \forall t \in I$.
$\Longrightarrow g(\alpha(t)) \leq g(\alpha(t_0)) \quad \forall t \in I$ or $g(\alpha(t)) \geq g(\alpha(t_0)) \quad \forall t \in I$.
$\Longrightarrow (g \circ \alpha)(t) \leq (g \circ \alpha)(t_0) \quad \forall t \in I$ or $(g \circ \alpha)(t) \geq (g \circ \alpha)(t_0) \quad \forall t \in I$.
$\Longrightarrow t_0$ is an extreme point of $g \circ \alpha$ on I.
Therefore,

$$\begin{aligned}
\frac{d}{dt}\left[(g \circ \alpha)(t_0)\right] &= 0 \\
\Longrightarrow \nabla g(\alpha(t_0)) \cdot \dot{\alpha}(t_0) &= 0 \\
\Longrightarrow \qquad \nabla g(p) \cdot v &= 0, \quad \forall v \in S_p \\
\Longrightarrow \nabla g(p) \in S_p^{\perp} &= [\nabla f(p)]^{\perp} \\
\Longrightarrow \qquad \nabla g(p) &= \lambda \nabla f(p)
\end{aligned}$$

■

Example 4. Show that the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, where $a, b, c \in \mathbb{R}$ on the unit circle $x_1^2 + x_2^2 = 1$ are eigenvalues of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Solution. Here we have given $S = f^{-1}(1)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2 + x_2^2$. Then

$$\nabla f(x_1, x_2) = (x_1, x_2, 2x_1, 2x_2)$$

and

$$\nabla g(x_1, x_2) = (x_1, x_2, 2ax_1 + 2bx_2, 2bx_1 + 2cx_2)$$

Let $p = (x_1, x_2) \in S$ be extreme point of g. Therefore by Lagrange multiplier theorem,

$$\nabla g(p) = \lambda \nabla f(p)$$

$\Longleftrightarrow$

$$\begin{aligned}
2ax_1 + 2bx_2 &= 2\lambda x_1 \\
2bx_1 + 2cx_2 &= 2\lambda x_2
\end{aligned}$$

or

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus the extreme points of g on S are eigenvectors of the symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of a matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$\begin{aligned} ax_1^2 + 2bx_1x_2 + cx_2^2 &= [x_1 \;\; x_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= [x_1 \;\; x_2]\, \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \lambda(x_1^1 + x_2^2) \\ &= \lambda \end{aligned}$$

Therefore, $g(p) = \lambda$, where $p = (x_1, x_2)$. Since a 2×2 matrix has only two eigenvalues, these eigenvalues are the maximum and minimum values of g on the compact set S.

Example 5. If $\mathbb{R}^4$ can be viewed as the set of all 2×2 matrices with real entries by identifying the 4-tuple (x_1, x_2, x_3, x_4) with matrix $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. The subset consisting of those matrices with determinant equal to 1 forms a group under matrix multiplication, called the special linear group SL(2). Show that SL(2) is 3-space in $\mathbb{R}^4$.

Solution. Here $SL(2) = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \text{ and } \begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix} = 1 \right\}$. Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in SL(2)$ hence $SL(2) \neq \phi$.

Also, $S = f^{-1}(1)$, where $f : \mathbb{R}^4 \to \mathbb{R}$ defined by $f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$ and $\nabla f(p) = (p, x_4, -x_3, -x_2, x_1)$, where $p = (x_1, x_2, x_3, x_4)$.

$$\nabla f(p) = 0 \iff (x_1, x_2, x_3, x_4) = 0$$

But $0 \notin SL(2)$ because determinant of zero-matrix is 0.

Therefore, SL(2) is 3-surface in $\mathbb{R}^4$.

Example 6. Let S be an $(n-1)$-surface in $\mathbb{R}^n$, given by $f^{-1}(c)$, where $f : U \to \mathbb{R}(U \text{ open in } \mathbb{R}^n)$ is such that $\nabla f(p) \neq 0$ for all $p \in f^{-1}(c)$. Let $g : U_1 \to \mathbb{R}$, where $U_1 = U \times \mathbb{R} = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, x_2, ..., x_n) \in U\}$ be defined by

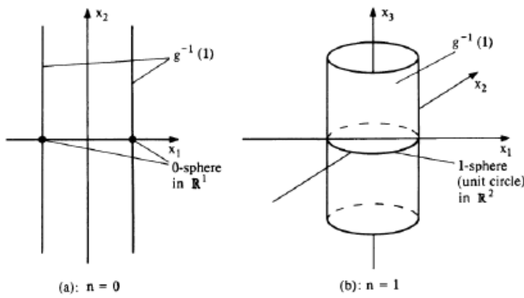
$$g(x_1, x_2, ..., x_{n+1}) = f(x_1, x_2, ..., x_n).$$

Then $g^{-1}(c)$ is an n-surface in R^{n+1}.

Solution. Since

$$\nabla g(x_1, x_2, ..., x_{n+1}) = \left(x_1, x_2, ..., x_{n+1}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}, 0 \right)$$

and $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, ..., \dfrac{\partial f}{\partial x_n}$ cannot be simultaneously zero when $g(x_1, x_2, ..., x_{n+1}) = f(x_1, x_2, ..., x_n) = c$ because $\nabla f(x_1, x_2, ..., x_n) \neq 0$, whenever $(x_1, x_2, ..., x_n) \in f^{-1}(c)$. The n-surface $g^{-1}(c)$

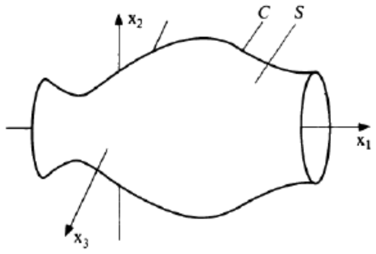


The cylinder $g^{-1}(1)$ over the n-sphere: $g(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_n^2$.

is called cylinder over S.

Example 7. The Surface of Revolution:

Let C be a curve in $\mathbb{R}^2$ which lies above x_1-axis. Thus $C = f^{-1}(c)$ for some $f : U \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in C$, where U is contained in the upper half plane $x_2 > 0$. Define $S = g^{-1}(c)$ where $g : U \times \mathbb{R} \to \mathbb{R}$ by $g(x_1, x_2, x_3) = f(x_1, (x_2^2 + x_3^2)^{1/2})$. Then S is 2-surface. Each point $(a, b) \in C$ generates a circle of point of S, namely circle in the plane $x_1 = a$ consisting of those points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 = a, x_2^2 + x_3^2 = b^2$. S is called surface of revolution obtained by rotating the curve C about the x_1-axis.



The surface of revolution S obtained by rotating the curve C about the x_1-axis.

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