Linear Algebra

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A non-empty set V is said to be a *vector space* over $\mathbb{R}$ (the set of real numbers) if there exist maps $+ : V \times V \to V$, defined by $(x, y) \mapsto x + y$, called *addition*, and $\cdot : \mathbb{R} \times V \to V$, defined by $(\alpha, y) \mapsto \alpha \cdot y$, called *scalar multiplication*, satisfying the following **eight** properties :

- $x + y = y + x, \ \forall \ x, y \in V$ (commutativity of addition).

- $(x + y) + z = x + (y + z), \ \forall\, x, y, z \in V$ (associativity of addition).
- There exists $0 \in V$ such that $x + 0 = x = 0 + x, \ \forall\, x \in V$ (existence of additive identity).
- For every $x \in V$ there exists $y \in V$ such that $x + y = 0 = y + x, \ \forall\, x, y \in V$.
  This y is denoted by $-x$.
  (existence of additive inverse).

Definition : Vector Space (continued...)

- $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \ \forall \ \alpha \in \mathbb{R}$ and $x, y \in V$.
- $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x, \ \forall \ \alpha, \beta \in \mathbb{R}$ and $x \in V$.
- $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x), \ \forall \ \alpha, \beta \in \mathbb{R}$ and $x \in V$.
- For $1 \in \mathbb{R}$, $1 \cdot x = x, \ \forall \ x \in V$.

For example, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{C}$ are all vector spaces over $\mathbb{R}$. Note that $\mathbb{Q}$ is not a vector space over $\mathbb{R}$.

Remark 1 :

1. Substraction : $x - y = x + (-y), \ \forall\, x, y \in V$.
2. Scalar multiplication :
   $\alpha x = \alpha \cdot x, \ \forall\, \alpha \in \mathbb{R}$ and $x \in V$.
3. $\mathbb{R}$ can be replaced by any **Field** (F) like $\mathbb{Q}, \mathbb{C}$, etc. In that case V is called vector space over F.
4. Elements of a vector space V are called vectors of V, and 0 is called the zero vector.
5. Sometimes, a vector space V can also be represented as a structure $< V, +, \cdot >$

Show that
$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, \ \forall\, i, \ 1 \leq i \leq n\}$
is a vector space under the addition and the scalar multiplication defined as follows.
For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$
and $\alpha \in \mathbb{R}$,
$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and
$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

Note that, the zero vector is $\mathbf{0} = (0, 0, \ldots, 0)$.

Example 2 : $M_{m\times n}(\mathbb{R})$ is a vector space.

Show that $M_{m\times n}(\mathbb{R})$ is a vector space under the addition and the scalar multiplication defined as follows.
For $A = [a_{ij}], B = [b_{ij}] \in M_{m\times n}(\mathbb{R})$ and $\alpha \in \mathbb{R}$,
$A + B = [a_{ij} + b_{ij}]$ and $\alpha A = [\alpha a_{ij}]$.

If $m = n$ then the set $M_{m\times n}(\mathbb{R})$ is denoted by $M(n, \mathbb{R})$ or $M_n(\mathbb{R})$.
Let S_n and A_n denotes the set of symmetric matrices and skew symmetric matrices respectively.

Note that $M(n, \mathbb{R})$, S_n and A_n are also vector spaces.

Let $S = \{(x_n) | x_n \in \mathbb{R}\}$ be the set of all real sequences.
Show that S is a vector space under the addition and the scalar multiplication defined as follows.
For $(x_n), (y_n) \in S$ and $\alpha \in \mathbb{R}$,
$(x_n) + (y_n) := (x_n + y_n)$ and $\alpha(x_n) := (\alpha x_n)$.

Let C be the set of all convergent sequences.
Let $C_0 = \{(x_n) | \lim_{n \to \infty} x_n = 0\}$.
Note that C and C_0 are also vector spaces, and $C_0 \subseteq C \subseteq S$.

Example 4 : $\mathcal{F}(X, \mathbb{R}) = \{f | f : X \to \mathbb{R}\}$ is a vector space.

Let X be a non-empty set.
Let $V = \mathcal{F}(X, \mathbb{R}) = \{f | f : X \to \mathbb{R}\}$ be the set of all real valued functions on the set X.
Show that V is a vector space under the addition and the scalar multiplication defined as follows.
For $f, g \in V$ and $\alpha \in \mathbb{R}$,
$(f + g)(x) = f(x) + g(x), \ \forall\, x \in X$, and
$(\alpha f)(x) = \alpha f(x), \ \forall\, x \in X$.

Let $\mathcal{C}([a, b]), \mathcal{D}([a, b])$ and $\mathcal{R}([a, b])$ be the set of all continuous, differentiable and Riemann integrable (real valued) functions defined on $[a, b]$. Then these are subsets of $\mathcal{F}([a, b], \mathbb{R})$ and are also vector spaces.

Remark 2 :

Note that, the above Example 4 is a generalized form of Ex. 1, Ex. 2 and Ex. 3 above, as it can easily be seen respectively as follows.

1. In Ex.1, take $X = \{1, 2, \ldots, n\}$ and define $f : X \to \mathbb{R}$ by $f(i) = x_i, \ \forall\ i, 1 \leq i \leq n$. Then the map $T : f \to (f(1), f(2), \ldots, f(n))$ is a bijection of $\mathcal{F}(X, \mathbb{R})$ and $\mathbb{R}^n$.
2. In Ex.2, take $X = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ and define $f : X \to \mathbb{R}$ by $f((i, j)) = a_{ij}, \ \forall\ i, j, 1 \leq i \leq m, 1 \leq j \leq n$. Then the map $T : f \to [a_{ij}]$ is a bijection of $\mathcal{F}(X, \mathbb{R})$ and $M_{m \times n}(\mathbb{R})$.
3. In Ex.3, take $X = \mathbb{N}$ and define $f : X \to \mathbb{R}$ by $f(i) = x_i, \ \forall\ i \in \mathbb{N}$. Then the map $T : f \to (x_i)$ is a bijection of $\mathcal{F}(X, \mathbb{R})$ and $S = \{(x_n) | x_n \in \mathbb{R}\}$.
4. $\mathcal{D}([a, b]) \subset \mathcal{C}([a, b]) \subset \mathcal{R}([a, b])$.

Example 5 : Set of polynomials $\mathcal{P}$ is a vector space.

Let $\mathcal{P} = \{\sum_{i=0}^{n} a_i x^i | a_i \in \mathbb{R}, n \in \mathbb{N}\}$ be the set of all polynomials in terms of variable x with real coefficients.
Show that $\mathcal{P}$ is a vector space under the addition and the scalar multiplication defined as follows.

For $p(x) = \sum_{i=0}^{m} a_i x^i, q(x) = \sum_{i=0}^{n} b_i x^i \in V$ and $\alpha \in \mathbb{R}$,

$p(x) + q(x) = \sum_{i=0}^{r} (a_i + b_i) x^i$, $r = max\{m, n\}$, and $\alpha p(x) = \sum_{i=0}^{m} (\alpha a_i) x^i$.

Let $\mathcal{P}_n = \{\sum_{i=0}^{n} a_i x^i | a_i \in \mathbb{R}\}$ be the set of all polynomials of degree $\leq n$.
Then $\mathcal{P}_n$ is a vector space. Note that the set of all polynomials exactly of degree n is not a vector space, since there does not exist a zero vector.

Direct sum of vector spaces :

Let V and W be vector spaces.
Define an addition and a scalar multiplication on the cartesian product $V \times W$ as follows.
For $(v_1, w_1), (v_2, w_2) \in V \times W$ and $\alpha \in \mathbb{R}$,
$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and
$\alpha(v_1, w_1) = (\alpha v_1, \alpha w_1)$.
Then $V \times W$ is a vector space, called **direct sum** of V and W, denoted by $V \oplus W$.
Note that $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$.

Theorem 1 : In a vector space V, we have

1. $0 \cdot x = \mathbf{0}$ for all $x \in V$.
2. There is a **unique** additive identity.
3. The additive inverse is **unique**.
4. $(-1) \cdot x = -x$ for all $x \in V$.
5. $\alpha \cdot \mathbf{0} = \mathbf{0}$ for all $\alpha \in \mathbb{R}$ and $\mathbf{0} \in \mathbf{V}$.
6. If $\alpha \cdot x = \mathbf{0}$ for $\alpha \in \mathbb{R}$ and $x \in V$, then either $\alpha = 0$ or $x = \mathbf{0}$.

1. Claim : $0 \cdot x = \mathbf{0}$ for all $x \in V$.
Note that $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$.
Now $\mathbf{0} = 0 \cdot x + (-0 \cdot x) = (0 \cdot x + 0 \cdot x) + (-0 \cdot x) = 0 \cdot x + (0 \cdot x + (-0 \cdot x)) = 0 \cdot x + \mathbf{0} = 0 \cdot x$.

2. Let $\mathbf{0}$ and $\mathbf{0}'$ be two additive identities of V.
Claim : $\mathbf{0} = \mathbf{0}'$.
As $x + \mathbf{0} = x = \mathbf{0} + x,\ \forall\, x \in V$ and also $x + \mathbf{0}' = x = \mathbf{0}' + x,\ \forall\, x \in V$, we have in particular, $\mathbf{0}' + \mathbf{0} = \mathbf{0}' = \mathbf{0} + \mathbf{0}'$ and $\mathbf{0} + \mathbf{0}' = \mathbf{0} = \mathbf{0}' + \mathbf{0}$.
Thus $\mathbf{0} = \mathbf{0}'$.

3. Let y and y' be two additive inverses of x in V.
Claim : $y = y'$.
We have $x + y = \mathbf{0} = y + x$ and also $x + y' = \mathbf{0} = y' + x$. Consider
$y = y + \mathbf{0} = y + (x + y') = (y + x) + y' = \mathbf{0} + y' = y'$.
Thus $y = y'$.
4. Consider
$(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = ((-1) + 1) \cdot x = 0 \cdot x = \mathbf{0}$.
Similarly $x + (-1) \cdot x = \mathbf{0}$. Thus $(-1) \cdot x = -x$.

Proof (Continued...)

5. Claim : $\alpha \cdot \mathbf{0} = \mathbf{0}$ for all $\alpha \in \mathbb{R}$ and $\mathbf{0} \in \mathbf{V}$.
Note that $\alpha \cdot \mathbf{0} = \alpha \cdot (\mathbf{0} + \mathbf{0}) = \alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0}$. Now
$\mathbf{0} = \alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0}) = (\alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0}) + (-\alpha \cdot \mathbf{0}) = \alpha \cdot \mathbf{0} + (\alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0})) = \alpha \cdot \mathbf{0} + \mathbf{0} = \alpha \cdot \mathbf{0}$

6. Claim : If $\alpha \cdot x = \mathbf{0}$ for $\alpha \in \mathbb{R}$ and $x \in V$, then either $\alpha = 0$ or $x = \mathbf{0}$
If $\alpha = 0$ then we are done. So suppose $\alpha \neq 0$.
Consider $\alpha \cdot x = \mathbf{0}$ $\quad \therefore \alpha^{-1} \cdot (\alpha \cdot x) = \alpha^{-1} \cdot \mathbf{0}$
$\therefore (\alpha^{-1}\alpha) \cdot x = \mathbf{0}$ $\quad \therefore 1 \cdot x = \mathbf{0}$ $\quad$ Thus $x = \mathbf{0}$.

Let W be a non-empty subset of a vector space V. Then W is said to be a vector subspace (or simply a subspace) of V if W itself is a vector space under the operations induced from V. That is,

1. $\mathbf{0} \in W$.
2. If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$.
3. If $\alpha \in \mathbb{R}$ and $w \in W$, then $\alpha w \in W$.

Theorem 2 : A subset W of V is a subspace of V if and only if
1. W is non-empty.
2. For $v, w \in W$ and for $\alpha, \beta \in \mathbb{R}$, $\alpha v + \beta w \in W$.

Examples :

1. $\mathbb{R}$ is a subspace of $\mathbb{C}$.
2. $\mathcal{D}([0,1])$ is a subspace of $\mathcal{C}([0,1])$.
3. $W = \{(a,b,0) | a,b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.
4. $\mathcal{P}_n$ is a subspace of $\mathcal{P}$.
5. S_n is a subspace of $M_n(\mathbb{R})$.
6. C is a subspace of S.
7. The set of bounded real valued functions is a subspace of $\mathcal{F}(X, \mathbb{R})$.

8. The set of all solutions of a homogeneous system $AX = O$ of linear equations in terms of n variables is a subspace of $\mathbb{R}^n$.

Note that, the set of all solutions of a non-homogeneous system $AX = B$ of linear equations in terms of n variables is **not** a subspace of $\mathbb{R}^n$ due to absence of a zero vector. State two more reasons.

Theorem 3 : If U and W are subspaces of V then $U \cap W$ is also a subspace of V.

What about $U \cup W$?

Note that, the union of X-axis and Y-axis is **not** a subspace of $\mathbb{R}^3$, since $(1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin U \cup W$.

Sum of two subspaces :

Let U and W be two subspaces of a vector space V. Then sum of U and W, denoted by $U + W$, is defined as $U + W = \{u + w | u \in U, w \in W\}$.

Theorem 4 : The sum $U + W$ of the subspaces U and W of V is also a subspace of V.

For example, let $U = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} | a, b \in \mathbb{R} \right\}$ and $W = \left\{ \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} | c, d \in \mathbb{R} \right\}$. Then $U + W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} | x, y, z \in \mathbb{R} \right\}$ is a subspace of $V = M_2(\mathbb{R})$.

Vector space as a direct sum of its two subspaces :

The vector space V is said to be the direct sum of its subspaces U and W, denoted by $U \oplus W$, if every vector v of V can be written in **unique** way as $v = u + w$, where $u \in U$ and $w \in W$.

Theorem 5 : The vector space V is the direct sum of its subspaces U and W if and only if $V = U + W$ and $U \cap W = \{\mathbf{0}\}$.

Ex. : $\mathbb{R}^3$ is the direct sum of XY-plane and Z-axis.

Note that, $\mathbb{R}^3$ is **not** the direct sum of XY-plane and YZ-plane,

since (3, 5, 7) = (3, 1, 0) + (0, 4, 7) = (3, -4, 0) + (0, 9, 7).

Note also that, $M_n(\mathbb{R}) = S_n \oplus A_n$, since any $X \in M_n(\mathbb{R})$ can be written as $X = Y + Z$, where $Y = (X + X^t)/2 \in S_n$ and $Z = (X - X^t)/2 \in A_n$.

Solve the following:

1. Show that $W = \{(a, b, c) | a + b + c = 0, a, b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.
2. Show that $W = \{(a, b, c) | a \geq 0, a, b, c \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^3$.
3. Show that $W = \{(a, b, c) | a^2 + b^2 + c^2 \leq 1, a, b, c \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^3$.

Let V be a vector space over $\mathbb{R}$.
Let $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of V.
Then any vector $v \in V$ of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ is called a **linear combination** of $v_1, v_2, \ldots, v_k$, where for each i, $\alpha_i \in \mathbb{R}$.

The set of all linear combinations of the vectors in S, denoted by $L(S)$, is the smallest subspace of V containing S, called the **linear span** of the set S.

In other words, $L(S)$ is the subspace spanned or generated by S, notationally, $L(S) = Span(S) =< S >$. We define $L(\emptyset) = \{\mathbf{0}\}$.

Example : Let $S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$. Then $\mathbb{R}^3 = L(S)$, since for any $(a,b,c) \in \mathbb{R}^3$, $(a,b,c) = ae_1 + be_2 + ce_3$.

Consider $A = [a_{ij}]_{m \times n}$.

Let $R_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, where $1 \leq i \leq m$. Clearly $R_i \in \mathbb{R}^n$. Let $S = \{R_1, R_2, \ldots, R_m\}$. Then $L(S)$ is subspace of $\mathbb{R}^n$, called the **row space** of A.

Similarly, one can define the **column space** of A, as the subspace of $\mathbb{R}^m$.

Note that, row equivalent matrices have the same row space.

Let V be a vector space over $\mathbb{R}$.

A vector $v \in V$ is said to be dependent on the vectors $v_1, v_2, \ldots, v_k \in V$ if there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$.

The vectors $v_1, v_2, \ldots, v_k \in V$ are said to be **linearly dependent** over $\mathbb{R}$, or simply dependent, if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, not all of them 0, such that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0}$. Otherwise, the vectors are said to be **linearly independent** over $\mathbb{R}$, or simply independent.

Thus, if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0}$ implies that $\alpha_i = 0$, for each i, $1 \leq i \leq k$, then the vectors $v_1, v_2, \ldots, v_k \in V$ are linearly independent.

Remark :

1. The set $S = \{v_1, v_2, \ldots, v_k\}$ is said to be linearly independent, if the vectors $v_1, v_2, \ldots, v_k$ are linearly independent.
   The empty set $\emptyset$ is defined to be independent.
2. If subset of a set is dependent then the set is also dependent. Hence any subset of an independent set is independent.
3. A non-zero vector is independent.
4. If any one of the vectors is zero or any two are same then the vectors are dependent.
5. Two vectors are dependent if and only if one of them is a scalar multiple of the other.

Show that the set $S = \{(1,1,1),(1,1,0),(1,0,0)\}$ is linearly independent. What is $L(S)$?
Solution : Consider
$k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,0) = (0,0,0)$.
That is, $(k_1 + k_2 + k_3, k_1 + k_2, k_1) = (0,0,0)$.
Therefore $k_1 + k_2 + k_3 = 0, k_1 + k_2 = 0, k_1 = 0$.
Thus $k_1 = 0, k_2 = 0, k_3 = 0$.
$L(S) = \mathbb{R}^3$, since for any $(a,b,c) \in \mathbb{R}^3$, there exist scalars $k_1 = c, k_2 = b - c, k_3 = a - b \in \mathbb{R}$ such that $(a,b,c) = k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,0)$.

Let V be a vector space over $\mathbb{R}$. Then

V is said to be a vector space of **dimension** n, denoted by $dim(V) = n$, if there exists a set B of linearly independent vectors $v_1, v_2, \ldots, v_k$ which span V. The set B is called a **basis** of V.

In other words, B is a basis of V if any $v \in V$ can be expressed **uniquely** as the linear combination of $v_1, v_2, \ldots, v_k$.
From above Example 1, S is a basis of $\mathbb{R}^3$, and so $\mathbb{R}^3$ is of dimension 3.

Note that, as $\emptyset$ is independent, the vector space $L(\emptyset) = \{\mathbf{0}\}$ is defined to have dimension 0. Also, when vector space is not of finite dimension, it is said to be of infinite dimension, for example, $\mathcal{P}$.

Let $B = \{e_1, e_2, \ldots, e_n\}$ be a basis of an n-dimensional vector space V over $\mathbb{R}$. Let $v \in V$. As B is a basis, v can be written *uniquely* as
$v = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$,
where for each i, $1 \leq i \leq n$, $\alpha_i \in \mathbb{R}$.

The vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called **coordinate vector** of v with respect to the basis B.
It is denoted by $[v]_B$ or simply by $[v]$.

In general $[v]$ depends not only on the basis (and the order of the elements in the basis) but also the field F over which V is defined.

Example 2 :

- Consider the basis set
  $S = \{(1,1,1),(1,1,0),(1,0,0)\}$ of $\mathbb{R}^3$.
  Let $v = (1,2,3) \in V = \mathbb{R}^3$. Then
  $[v]_S = (3,-1,-1)$.
- Now consider the standard basis set
  $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$.
  Let $v = (1,2,3) \in \mathbb{R}^3$. Then $[v]_B = (1,2,3)$.
- Also, if we consider the set
  $B' = \{(1,2,-3),(1,-3,2),(2,-1,5)\}$. Then
  B' is a basis of $\mathbb{R}^3$, and for $v = (1,2,3) \in \mathbb{R}^3$,
  $[v]_{B'} = (0,-1,1)$.

The set of all solutions of a homogeneous system $AX = O$ of m linear equations in terms of n variables is a vector subspace of $\mathbb{R}^n$, called the **null (or solution) space** of A. The dimension of null space of A is called **nullity** of A, denoted by $\eta(A)$.

The set of all vectors $Y \neq O$ such that $AX = Y$ for some $X \in \mathbb{R}^n$ is a vector subspace of $\mathbb{R}^m$, called the **range space** of A. The dimension of range space of A is called **rank** of A, denoted by $\rho(A)$.

Remark : By definition of the range space of A, that is, using $AX = Y$, it can be concluded that each Y can be written as the linear combination of the columns of A with scalars, precisely entries of X.
Therefore the range space of A is spanned by the column vectors of A.
Hence the rank of A is nothing but the number of linearly independent column vectors of A.
Thus, the set of these linearly independent vectors forms the basis of the range space of A.

Theorem : If A is an $m \times n$ matrix then $\rho(A) + \eta(A) = n$.

Verify above Theorem for $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$.

Remark :

- Let S be a set with two or more vectors in a vector space V. Then S is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the rest of the vectors in S.
- A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.
- Geometrically, a set of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.
  Also, a set of three vectors in $\mathbb{R}^3$ is linearly independent if and only if the vectors do not lie on the same plane when they are placed with their initial points at the origin.
- If U, W are subspaces of a vector space V then $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$.

- If $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of a vector space V, then every set with more than n vectors of V is linearly dependent.
- If V is a finite dimensional vector space, then any two basis sets of V have the same number of vectors.
- If V is an n dimensional vector space then
(1) Any set with n linearly independent vectors in V is a basis of V.
(2) Any set with n vectors which spans V is a basis of V.
(3) If S is a linearly independent subset of V with $|S| < n$, then S can be enlarged to a basis set of V.
(4) If W is a subspace of V then $dimW \leq dimV$. Moreover, $dimW = dimV$ if and only if $W = V$.
- The standard basis of $\mathcal{P}_n$ is the set $B = \{1, x, x^2, x^3, \ldots, x^n\}$.

Let V be a finite dimensional vector space. A linearly independent subset S of V is said to be **maximal linearly independent set** of V, if $S \cup \{v\}$ is dependent, for any vector $v \in V$. A generating subset S of V is said to be **minimal generating set** of V, if $S \setminus \{u\}$ is not a generating set, for any vector $u \in S$.

Theorem : Let V be a finite dimensional vector space. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a subset of V. Then the following statements are equivalent.

1. B is a basis of V.
2. B is a maximal linearly independent set.
3. B is a minimal generating set.

Exercise :

- Let $V = \mathbb{R}^+$. For $x, y \in V$ and for $\alpha \in \mathbb{R}$, define $x + y = x \cdot y$ and $\alpha x = x^\alpha$.
  Show that V is a vector space over $\mathbb{R}$.
- Show that $W = \{f | f(1) = 0\}$ is a subspace of the vector space V of all real valued functions.
- Check whether the matrices $\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$ are linearly independent.

- Show that $e^x, \sin x$ and $\cos x$ are linearly independent in the vector space V of all real valued functions.
- Show that the polynomials $1 - x, 1 + 3x - x^2$ and $5 + 3x - 2x^2$ are linearly dependent in $\mathcal{P}_2$.
- Show that $\{x, 3x^2, x + 5\}$ forms a basis of $\mathcal{P}_2$.
- Let $W = \{(x, y, z, w) | y + z = 0, x = 2w\}$.
  Prove that W is a subspace of $\mathbb{R}^4$.
  Also, find a basis and the dimension of W.

- Find the rank and the nullity of the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 5 & 1 \\ 3 & 5 & 8 \end{bmatrix}$$

- Find a basis and the dimension of the solution space of the following system of linear equations. $x + 2y + 7z = 0, -2x + y - 4z = 0, x - y + z = 0.$
- Let S and T be subsets of the vector space V. Then prove that
  (1) If $S \subset T$ then $L(S) \subset L(T)$.
  (2) $L(S) = S$ if and only if S is a subspace of V.
  (3) $L(L(S)) = L(S)$.

Thank you